SYMMETRIC DIAMOND WAVES REVISITED: EXISTENCE VIA THE CRANDALL-RABINOWITZ THEOREM

A Thesis

presented to

the Faculty of the Graduate School

at the University of Missouri-Columbia

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

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Dr. Samuel Walsh, Thesis Supervisor

MAY 2026

The undersigned, appointed by the dean of the Graduate School, have examined the thesis entitled

SYMMETRIC DIAMOND WAVES REVISITED: EXISTENCE VIA THE CRANDALL-RABINOWITZ THEOREM

presented by Krishna Chebolu,

a candidate for the degree of Master of Science in Applied Mathematics,

and hereby certify that, in their opinion, it is worthy of acceptance.

Professor Samuel Walsh

Professor Konstantin Makarov

Professor Tanya Christiansen

DEDICATION

To my brothers, Rohit and Rahul —

conquerors of thirst, champions of the arid morning,

May my study of water inspire you.

To my family and friends, whose support and encouragement

carried me through every challenge.

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my thesis supervisor, Professor Samuel Walsh, whose guidance, patience, and continual support made this work possible. From suggesting the problem to carefully reading many drafts line-by-line, Dr. Walsh was an extraordinary mentor throughout every stage of this project. His mathematical insight, encouragement, and willingness to meet and discuss ideas at length shaped not only this thesis, but my development as a mathematician. I am truly grateful for the time and care he invested in my education.

I would also like to thank the members of my thesis committee, Professor Konstantin Makarov and Professor Tanya Christiansen, for their time and willingness to serve on my committee and for their careful reading of this work.

Finally, I am grateful to the Department of Mathematics at the University of Missouri for providing a supportive and stimulating environment in which to pursue this degree.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SYMMETRIC DIAMOND WAVES REVISITED: EXISTENCE VIA THE CRANDALL-RABINOWITZ THEOREM

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ABSTRACT

The three-dimensional capillary-gravity water wave problem models the motion of an inviscid, incompressible fluid bounded by a free surface and a flat bottom. This thesis investigates the rigorous existence of traveling wave solutions that are doubly periodic in the horizontal directions, giving rise to symmetric surface patterns known as diamond waves. Reeder and Shinbrot in [1] first established these waves' existence via an ad hoc iteration scheme. In this thesis, we approach the problem through a more standard functional-analytic framework. Using the Zakharov–Craig–Sulem formulation [2, 3], we restrict the Euler equations to a coupled system on the free surface via the Dirichlet–Neumann operator, we apply the Crandall–Rabinowitz bifurcation theorem [4] to establish the emergence of non-trivial wave branches from the flat-water state. Furthermore, through the explicit computation of the third-order Fréchet derivatives and non-degeneracy conditions, we demonstrate that these nonlinear diamond waves emerge via a supercritical pitchfork bifurcation.

Chapter 1

INTRODUCTION

1.1 Brief history and outline

The mathematical study of water waves has a long and rich history, dating back to the foundational work of Laplace, Lagrange, and Cauchy in the late eighteenth and early nineteenth centuries. It is among the oldest and most physically motivated problems in mathematical physics. The water wave problem describes the motion of a free surface of fluid under the influence of gravity and, in some formulations, surface tension. Despite its classical origins, it remains an active area of research, particularly concerning the existence and qualitative properties of special solutions such as traveling waves.

The simplest traveling wave solutions are the planar Stokes waves: periodic, two-dimensional waves that propagate in a single horizontal direction without changing shape. Their rigorous existence was first established by Nekrasov [5], Levi–Civita [6], and Struik [7] in the 1920s. The three-dimensional problem is considerably more analytically demanding. In the absence of surface tension, the linearized operator generically possesses an infinite-dimensional kernel, as infinitely many wave vectors may satisfy the dispersion relation at a given speed. This is the small-divisor phenomenon, which renders classical perturbative methods and the standard implicit function theorem inapplicable, and necessitates the use of Nash–Moser-type iteration schemes [8, 9]. When one seeks traveling waves that are periodic in both horizontal directions, the geometry of the wave pattern is governed by the choice of the underlying spatial lattice, giving rise to a variety of distinct surface patterns. Among these, diamond waves, that arise from the interaction of two wave trains of equal speed traveling in different directions, seem to us a particularly natural and physically compelling class of solutions. The first rigorous existence result for such waves was obtained by Reeder and Shinbrot [1], via an ad hoc iteration scheme.

This thesis revisits this existence result using more standard functional-analytic machinery. Specifically, we cast the water wave problem as an operator equation and establish the existence of diamond waves via the Crandall–Rabinowitz bifurcation theorem, replacing the ad hoc iteration of Reeder–Shinbrot. The remainder of this thesis is organized as follows. The rest of Chapter 1 sets up the mathematical formulation and introduces our problem of interest in more detail. Chapter 2 contains the necessary background material, presented as a collection of definitions and theorems. Chapter 3 contains the proof of the main theorem. In Chapter 4, we go beyond the existence result by proving the invertibility of the linearized operator along the bifurcating branch, and we characterize the type of bifurcation.

1.2 Mathematical setup

The capillary-gravity water wave problem concerns the motion of a body of water with a free surface, acted on by gravity and surface tension. We model the fluid as inviscid and incompressible, occupying a time-dependent domain

$$\Omega(t) = \{(\mathbf{x}, z) \in \mathbb{R}^2 \times \mathbb{R} : -d < z < \eta(\mathbf{x}, t)\}, \tag{1.1}$$

where $\mathbf{x} = (x, y)$ are the horizontal coordinates, z is the vertical coordinate, $d > 0$ is the undisturbed depth of the fluid, and $\eta(\mathbf{x}, t)$ is the unknown free surface (see Figure 1.1). The fluid domain is bounded below by a flat rigid bed at $z = -d$ and above by the free surface $S(t) := \{z = \eta(\mathbf{x}, t)\}$. We assume that the density ρ is constant throughout and set $\rho = 1$.

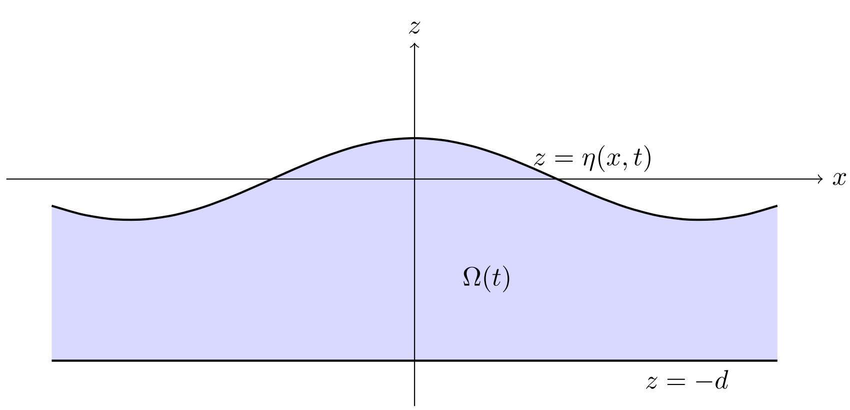

Figure 1.1: A 2D schematic of the water wave domain $\Omega(t)$ with free surface $z = \eta(x, t)$ and flat bottom $z = -d$.

The Euler equations. Since the fluid is inviscid and incompressible, the motion is governed by the Euler equations given below.

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P - g\mathbf{e}_3 \qquad \text{in } \Omega(t), \tag{1.2}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega(t). \tag{1.3}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, P is the pressure, $g > 0$ is the gravitational constant, and $\mathbf{e}_3$ is the unit vector in the vertical (z) direction. Equation (1.2) expresses conservation of momentum, and (1.3) is the incompressibility condition, which expresses the conservation of volume.

Boundary conditions. At the flat bed $z = -d$, the fluid cannot penetrate the boundary, so the normal velocity vanishes:

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } z = -d, \tag{1.4}$$

where $\mathbf{n}$ is the outward unit normal. At the flat bed $\mathbf{n} = -\mathbf{e}_3$, so this condition reduces to $u_3 = 0$. At the free surface $S(t)$, two conditions must hold. The *kinematic condition* states that fluid particles on the surface remain on the surface,

$$\partial_t \eta = \mathbf{u} \cdot \mathbf{n} \quad \text{on } S(t), \tag{1.5}$$

where $\mathbf{n} = (-\nabla' \eta, 1)$ is a non-unit outward normal to $S(t)$ and $\nabla' = (\partial_x, \partial_y)$. To see this, recall that $S(t)$ is the level set $F = z - \eta(\mathbf{x}, t) = 0$, and the gradient of F with respect to $(\mathbf{x}, z)$ is $\nabla_{\mathbf{x},z} F = (-\nabla' \eta, 1)$, which is always outward since increasing z points away from the fluid domain.

The *dynamic condition* balances the pressure jump across the free surface with the surface tension force. By the Laplace–Young equation, we can express this as follows

$$P = \sigma H(\eta) \quad \text{on } S(t), \tag{1.6}$$

where $\sigma > 0$ is the surface tension coefficient and

$$H(\eta) = \nabla' \cdot \left(\frac{\nabla' \eta}{\sqrt{1 + |\nabla' \eta|^2}} \right) \tag{1.7}$$

is twice the mean curvature of the free surface.

The irrotational case. We further assume that the flow is irrotational: $\nabla \times \mathbf{u} = 0$. Under this assumption $\mathbf{u} = \nabla\phi$ for a scalar velocity potential ϕ, and the incompressibility condition becomes Laplace's equation,

$$\Delta\phi = 0 \quad \text{in } \Omega(t). \tag{1.8}$$

The bed condition becomes $\partial_z\phi = 0$ at $z = -d$. The kinematic and dynamic conditions at the free surface become

$$\partial_t\eta - \partial_z\phi + \nabla'\eta \cdot \nabla'\phi = 0 \quad \text{on } S(t), \tag{1.9}$$

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g\eta - \sigma H(\eta) = 0 \quad \text{on } S(t). \tag{1.10}$$

Travelling waves. We seek solutions that are steady in a frame moving with speed c in the x-direction, so $\mathbf{c} = (c, 0)$. Setting

$$\eta = \eta(x - ct,\, y), \qquad \phi = \phi(x - ct,\, y, z), \tag{1.11}$$

the substitution $\partial_t \mapsto -c\partial_x$ reduces the system to the steady free boundary problem:

$$\Delta\phi = 0 \quad \text{in } \Omega, \tag{1.12}$$

$$\partial_z\phi = 0 \quad \text{on } z = -d, \tag{1.13}$$

$$-c\partial_x\eta - \partial_z\phi + \nabla'\eta \cdot \nabla'\phi = 0 \quad \text{on } S, \tag{1.14}$$

$$-c\partial_x\phi + \frac{1}{2}|\nabla\phi|^2 + g\eta - \sigma H(\eta) = 0 \quad \text{on } S, \tag{1.15}$$

where $\Omega = \{(\mathbf{x}, z) : -d < z < \eta(\mathbf{x})\}$ is now time independent.

1.3 Our problem

The setting of this thesis is the three-dimensional water wave problem described in the preceding section. We seek traveling wave solutions that are doubly periodic in the two horizontal directions and whose free surface forms a symmetric diamond pattern. The existence of such waves, aptly called *diamond waves* (see Figure 1.2), is the central result of this thesis. The first rigorous construction was due to Reeder and Shinbrot [1]. Instead of the iterative scheme provided in their paper, we give a standard version of their argument using the Crandall–Rabinowitz theorem [4].

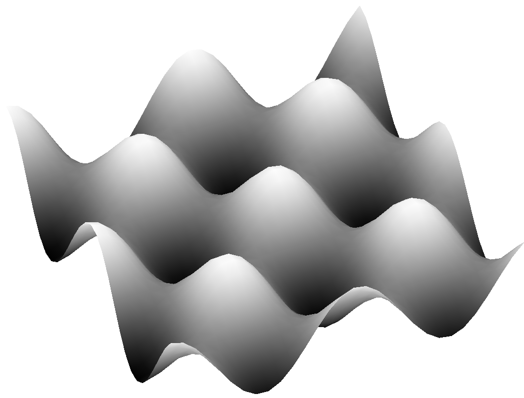

Figure 1.2: A doubly periodic (diamond) wave.

Periodicity and symmetry. We assume that solutions are periodic with respect to a two-dimensional horizontal lattice

$$\Lambda := \{m_1\mathbf{b}_1 + m_2\mathbf{b}_2 : m_1, m_2 \in \mathbb{Z}\} \subset \mathbb{R}^2, \tag{1.16}$$

where $\mathbf{b}_1, \mathbf{b}_2$ are two linearly independent vectors in the horizontal plane, see figure 1.3.

Any Λ-periodic function f admits a Fourier series expansion

$$f(\mathbf{x}) = \sum_{\mathbf{k}\in\Lambda^*} \hat{f}_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{1.17}$$

indexed by the dual lattice

$$\Lambda^* := \{\mathbf{k} \in \mathbb{R}^2 : \mathbf{k} \cdot L \in 2\pi\mathbb{Z} \text{ for all } L \in \Lambda\}, \tag{1.18}$$

where $\hat{f}_{\mathbf{k}}$ are the Fourier coefficients of f. The dual lattice Λ^* thus plays the role of the frequency domain: each $\mathbf{k} \in \Lambda^*$ corresponds to a single oscillatory mode $e^{i\mathbf{k}\cdot\mathbf{x}}$, and the full solution is built by superposing these modes. This Fourier-analytic structure is particularly convenient for our problem, as differential operators act diagonally on individual modes.

The diamond wave pattern arises when the lattice is chosen so that there exist two wave vectors $\mathbf{k}_1 = (\kappa_1, \kappa_2)$ and $\mathbf{k}_2 = (\kappa_1, -\kappa_2)$ in Λ^* with $|\mathbf{k}_1| = |\mathbf{k}_2|$. We write $\mathbf{k}_1 = |\mathbf{k}_1|(\cos\theta, \sin\theta)$, where $\theta \in (0, \pi/2)$ is the angle that $\mathbf{k}_1$ makes with the x-axis. To match the symmetry of this pattern, we restrict to the *symmetric subspace*, requiring η to be even in both x and y, and Φ to be odd in x and even in y. These parity conditions are consistent with the structure of $\mathcal{F}$ (1.25): the kinematic equation maps H^s_{oe} to H^{s-1}_{oe}, and the dynamic equation maps H^s_{ee} to H^{s-2}_{ee}, as can be verified by

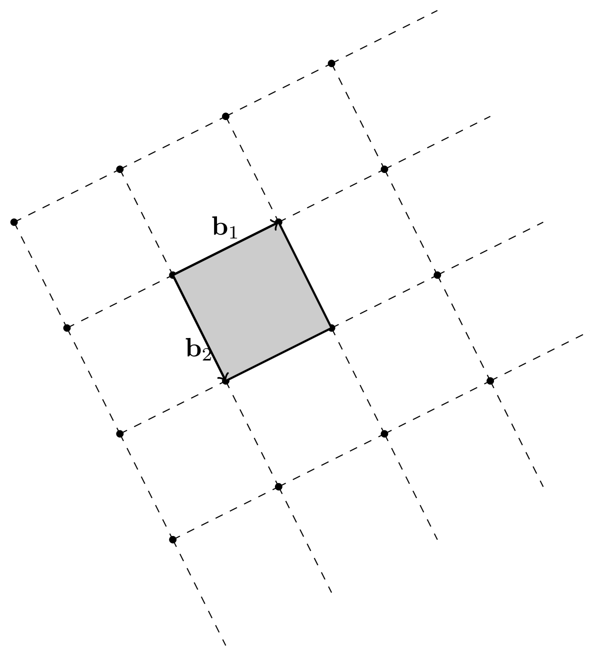

Figure 1.3: A representative image of the lattice. The points can be imagined to be the tops of crests or the bottoms of troughs.

examining each term. We therefore define the solution and target spaces, for $s > 2$, by

$$X := H^s_{\mathrm{ee}}(\Lambda) \times H^s_{\mathrm{oe}}(\Lambda), \qquad Y := H^{s-1}_{\mathrm{oe}}(\Lambda) \times H^{s-2}_{\mathrm{ee}}(\Lambda), \tag{1.19}$$

where

$$H^s_{\mathrm{ee}}(\Lambda) := \{u \in H^s(\Lambda) : u(-x,y) = u(x,y),\ u(x,-y) = u(x,y)\}, \tag{1.20}$$

$$H^s_{\mathrm{oe}}(\Lambda) := \{u \in H^s(\Lambda) : u(-x,y) = -u(x,y),\ u(x,-y) = u(x,y)\}. \tag{1.21}$$

Reduction to surface variables. The full water wave problem involves finding both the free surface η and the velocity potential ϕ throughout the interior fluid domain Ω. A classical reduction due to Zakharov [2], and made rigorous by Craig and Sulem [3], reformulates the problem entirely in terms of two surface quantities: the free surface height $\eta(\mathbf{x})$ and the surface potential $\Phi(\mathbf{x}) := \phi(\mathbf{x}, \eta(\mathbf{x}))$, which is the restriction of the velocity potential on the free surface. The key tool in this reduction is the Dirichlet–Neumann operator $G(\eta)$, defined by

$$G(\eta)\Phi := (\partial_z \phi - \nabla'\eta \cdot \nabla'\phi)\big|_{z=\eta(\mathbf{x})}, \tag{1.22}$$

which maps Dirichlet data Φ on the free surface to the scaled outward normal derivative of ϕ there. Using this operator, the full free boundary problem reduces to the system

$$c\,\partial_x\eta = G(\eta)\Phi, \tag{1.23}$$

$$c\,\partial_x\Phi = -g\eta + \sigma H(\eta) - \frac{1}{2}|\nabla'\Phi|^2 + \frac{\big(\nabla'\eta\cdot\nabla'\Phi + G(\eta)\Phi\big)^2}{2(1+|\nabla'\eta|^2)}, \tag{1.24}$$

posed entirely on $\mathbb{R}^2$.

Traveling waves and the operator equation. We seek solutions that travel at constant speed c in the x-direction, so that η and Φ depend on $(x-ct, y)$ rather than on t and x separately. Passing to the moving frame via $\partial_t \mapsto -c\partial_x$, the system becomes a steady nonlinear PDE problem in the horizontal variables (x, y). We define the nonlinear operator

$$\mathcal{F}(\eta,\Phi,\, c) := \begin{pmatrix} -c\,\partial_x\eta + G(\eta)\Phi \\ -c\,\partial_x\Phi + \dfrac{1}{2}|\nabla\Phi|^2 - \dfrac{\big(\nabla\eta\cdot\nabla\Phi + G(\eta)\Phi\big)^2}{2(1+|\nabla\eta|^2)} + g\eta - \sigma H(\eta) \end{pmatrix}, \tag{1.25}$$

and seek solutions to the operator equation

$$\mathcal{F}(\eta,\Phi,\, c) = 0. \tag{1.26}$$

The trivial solution $(\eta,\Phi) = (0,0)$ corresponds to undisturbed flat water and satisfies (1.26) for every wave speed c. We will look for nontrivial solutions bifurcating from this trivial branch. With the spaces X and Y defined above, $\mathcal{F} : X \times \mathbb{R} \to Y$ is a well-posed smooth operator in a neighborhood of the trivial solution, a consequence of the analyticity of $G(\eta)$, discussed in 2.3.

The nondegeneracy condition. The condition $(\lambda,\theta) \notin \mathscr{M}_d$, where $\lambda = \frac{1}{|\mathbf{k}_1|}\sqrt{\frac{g}{\sigma}}$ and $\mathscr{M}_d \subset (0,\infty)\times(0,\pi/2)$ is a nowhere dense set comprising the countable union of monotone curves, is a nondegeneracy condition on the physical parameters. It ensures that the only wave vectors in Λ^* satisfying the linear dispersion relation at the critical speed c^* are $\pm\mathbf{k}_1$ and $\pm\mathbf{k}_2$, preventing additional resonances from enlarging the kernel of the linearized operator. This is will be used to obtain the one-dimensionality of the kernel required by the Crandall–Rabinowitz theorem, refer to Appendix A.

The main result. With the setup above, the main theorem of this thesis is the following existence result for diamond waves. This theorem appears in [10, Thm 8.1], which is originally stated in Reeder and Shinbrot [1, Thm 6.5]. We will prove this theorem in Chapter 3.

Theorem 1. *Let* $\lambda = \frac{1}{|\mathbf{k}_1|}\sqrt{\frac{g}{\sigma}}$ *and write* $\mathbf{k}_1 = |\mathbf{k}_1|(\cos\theta, \sin\theta)$*. Assume that* $(\lambda, \theta) \notin \mathscr{M}_d$*, where* $\mathscr{M}_d \subset (0, \infty) \times (0, \pi/2)$ *is a nowhere dense set comprising the countable union of monotone curves. Then there exists a family of* Λ*-periodic, symmetric, smooth solutions* $(\eta, \Phi, c)(\varepsilon)$ *to* (1.26) *bifurcating at* $(\eta, \Phi, c) = (0, 0, c^*)$*, where*

$$c^* = c_p(\mathbf{k}_1) = c_p(\mathbf{k}_2), \qquad c_p(\mathbf{k})^2 = \frac{(g + \sigma|\mathbf{k}|^2)\,|\mathbf{k}|\tanh(|\mathbf{k}|d)}{\kappa_1^2}, \tag{1.27}$$

is the common speed of the two wave vectors. The solutions satisfy

$$\eta(\varepsilon) = \varepsilon\cos(\kappa_1 x)\cos(\kappa_2 y) + O(\varepsilon^2) \tag{1.28}$$

as $\varepsilon \to 0$*.*

The existence result of Theorem 1 describes the bifurcating branch at $c = c^*$, but does not address the geometry of the resulting curve. We complement it with the new following result, proved in Chapter 4.

Theorem 2. *Let the assumptions of Theorem 1 hold, and let* $(\eta(s), \Phi(s), c(s))$ *denote the bifurcating branch of diamond wave solutions. Then the following hold.*

(i) *The linearized operator* $L(s) := D_{(\eta,\Phi)}\mathcal{F}(\eta(s), \Phi(s), c(s))$ *is invertible for all* $0 < |s| \ll 1$*.*

(ii) *The resulting bifurcation is a supercritical pitchfork.*

Chapter 2

BACKGROUND

2.1 The Fréchet derivative

The Fréchet derivative is the generalized notion of differentiability for maps between Banach spaces. In finite dimensions, the derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point x is the linear map that best approximates f near x, in the sense that the error in the approximation $f(x + h) \approx f(x) + Df(x)h$ is $o(|h|)$ as $h \to 0$. The Fréchet derivative extends this idea to maps between infinite-dimensional Banach spaces, replacing the Jacobian matrix with a bounded linear operator and the Euclidean norm with the norms on the respective spaces. The key point is that the approximation must be uniform in direction: the error must go to zero faster than $\|h\|$ regardless of the direction in which h approaches zero, which is a stronger requirement than, say, the existence of directional derivatives alone.

In our setting, the nonlinear operator $\mathcal{F} = \mathcal{F}(\eta, \Phi, c)$ defined in (1.25) maps between the Sobolev spaces X and Y introduced in Section 1.3. Its Fréchet derivative at the trivial solution $(\eta, \Phi) = (0, 0)$ is the linearized operator L_c, whose kernel structure is a central object of the bifurcation analysis carried out in Chapter 3.

Definition 3. *Let X and Y be Banach spaces and let $\mathcal{F} \in C(X; Y)$ be given. We say that $\mathcal{F}$ is Fréchet differentiable at $x \in X$ if there exists a bounded linear operator $\mathcal{L}^x : X \to Y$ such that*

$$\lim_{h \to 0} \frac{\|\mathcal{F}(x + h) - \mathcal{F}(x) - \mathcal{L}^x(h)\|_Y}{\|h\|_X} = 0,$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms on X and Y respectively. We denote $(D\mathcal{F})(x) := \mathcal{L}^x$ or, sometimes, $D_x\mathcal{F}(x)$. If $\mathcal{F}$ is Fréchet differentiable at every $x \in X$, we say that $\mathcal{F}$ is Fréchet differentiable.

2.2 Bifurcation theory

We study how the solution set of an equation $\mathcal{F}(\lambda, x) = 0$ changes as a parameter λ varies. In many problems, a trivial solution exists for all values of λ, and one seeks parameter values at which nontrivial solutions emerge from this trivial state. The central tool for detecting such behavior is the structure of the linearized operator $D_x\mathcal{F}(\lambda, 0)$. When its kernel is nontrivial, the implicit function theorem fails and new solutions may bifurcate. Making this rigorous requires the use of Fredholm operators, which we now introduce.

Definition 4. *Let X and Y be Banach spaces and suppose that $\mathcal{L} : X \to Y$ is a bounded linear operator. Suppose that*

$$\dim \mathscr{N}(\mathcal{L}) < \infty, \qquad \operatorname{codim} \mathscr{R}(\mathcal{L}) < \infty,$$

where $\mathscr{N}(\mathcal{L})$ is the null space and $\mathscr{R}(\mathcal{L})$ is the range. If we also know that the range is a closed set in Y, then we say that $\mathcal{L}$ is a Fredholm operator*. Further, we define the* index *of $\mathcal{L}$, denoted* $\operatorname{ind} \mathcal{L}$*, to be the integer*

$$\operatorname{ind} \mathcal{L} := \dim \mathscr{N}(\mathcal{L}) - \operatorname{codim} \mathscr{R}(\mathcal{L}).$$

The condition $\operatorname{codim} \mathscr{R}(\mathcal{L}) < \infty$ means that the range misses only a finite-dimensional subspace of Y, in the sense that Y decomposes as $Y = \mathscr{R}(\mathcal{L}) \oplus Z$ for some finite-dimensional subspace $Z \subset Y$. In general, a closed subspace of a Banach space need not admit a closed complement, so such a decomposition cannot be taken for granted. Here, it is possible because Z is finite-dimensional, then every finite-dimensional subspace of a Banach space is automatically closed and complemented, so one can always write Y as a direct sum in this way.

In our case, the linearized operator $L_c := D_{(\eta,\Phi)}\mathcal{F}(0, 0, c)$ will be shown to be Fredholm of index zero, with a one-dimensional kernel at the critical speed c^*. This is in accordance with the setting of the following theorem, due to Crandall and Rabinowitz [4], which is our primary tool.

Theorem 5 (Crandall–Rabinowitz)**.** *Let X and Y be Banach spaces and let $\mathcal{F} : \mathbb{R} \times X \to Y$ be a C^2 map. Suppose that:*

(i) $\mathcal{F}(\lambda, 0) = 0$ *for all* $\lambda \in \mathbb{R}$*;*

(ii) *there exists* $\lambda_0 \in \mathbb{R}$ *such that* $D_x\mathcal{F}(\lambda_0, 0)$ *is a Fredholm operator of index zero with* $\dim \mathscr{N}(D_x\mathcal{F}(\lambda_0, 0)) = 1$*, say* $\mathscr{N}(D_x\mathcal{F}(\lambda_0, 0)) = \operatorname{span}\{u_0\}$*;*

(iii) *the transversality condition holds:*

$$D_\lambda D_x \mathcal{F}(\lambda_0, 0) u_0 \notin \mathscr{R}(D_x \mathcal{F}(\lambda_0, 0)).$$

Then $(\lambda_0, 0)$ *is a bifurcation point. More precisely, there exists* $\varepsilon > 0$ *and a* C^1 *curve*

$$\mathscr{C} = \{(\lambda(s), x(s)) : s \in (-\varepsilon, \varepsilon)\} \subset \mathbb{R} \times X,$$

with $(\lambda(0), x(0)) = (\lambda_0, 0)$*, such that* $\mathcal{F}(\lambda(s), x(s)) = 0$ *for all* $s \in (-\varepsilon, \varepsilon)$ *and* $x(s) \neq 0$ *for* $s \neq 0$*. Moreover, as* $s \to 0$*, the curve admits the asymptotic expansions*

$$\lambda(s) = \lambda_0 + O(s), \qquad x(s) = s u_0 + O(s^2),$$

so that for $|s| \ll 1$*, the nontrivial solutions are small perturbations of the kernel element* u_0*, branching off the trivial solution at the critical parameter value* λ_0*.*

For $s \neq 0$, the curve $\mathscr{C}$ consists entirely of nontrivial solutions: the state $x(s) = su_0 + O(s^2)$ is a perturbation of the flat rest state, parameterized by s. Then $\lambda(s) = \lambda_0 + O(s)$ describes how the bifurcation parameter must be adjusted to sustain the nontrivial solution at s. In this way, the theorem guarantees not just the existence of a single nontrivial solution, but a continuous one-parameter family of them, parametrized by s, and bifurcating from the trivial state at our critical point $\lambda = \lambda_0$. In our setting, λ plays the role of the wave speed c, $u_0 = \zeta^*$ is the kernel generator identified in Lemma 8, and s is ε.

2.3 The Dirichlet–Neumann operator

This operator enables us to reformulate the water wave problem as a system posed entirely on the free surface. Recall from Section 1.3 that given the free surface height η and the restriction Φ, the velocity potential ϕ in the interior fluid domain Ω is the unique harmonic function satisfying $\phi|_{z=\eta} = \Phi$ and $\partial_z \phi|_{z=-d} = 0$. The Dirichlet–Neumann operator $G(\eta)$ then maps the Dirichlet data Φ to the (scaled) outward normal derivative of ϕ at the free surface, encoding all of the interior fluid dynamics in a single surface operator. This reduction is what allows the full three-dimensional free boundary problem to be studied as a system in the two horizontal variables alone.

For the proofs in Chapters 3 and 4, three properties of $G(\eta)$ are essential. First, $G(\eta)$ depends analytically on η as a map on Sobolev spaces. Lannes presented this analyticity result in [11, Thm 3.15], building on the earlier work of Craig and Sulem [3]. This guarantees that $\mathcal{F}$ is smooth near the trivial solution and that the Fréchet derivatives computed in Chapters 3 and 4 are justified.

Second, the analyticity of $G(\eta)$ allows us to linearize the Dirichlet–Neumann term at the trivial solution explicitly. Since the term $D_\eta G(0) \cdot \dot{\eta}$ is evaluated at $\Phi = 0$, it vanishes, and the linearization reduces to

$$D_{(\eta,\Phi)}[G(\eta)\Phi]\big|_{(0,0)}(\dot{\eta}, \dot{\Phi}) = [D_\eta G(0) \cdot \dot{\eta}] \cdot 0 + G(0)\dot{\Phi} = G_0\dot{\Phi}, \tag{2.1}$$

where $G_0 := G(0)$ is the flat-surface Dirichlet–Neumann operator. At $\eta = 0$, this operator acts as a Fourier multiplier,

$$G_0 e^{i\mathbf{k}\cdot\mathbf{x}} = |\mathbf{k}| \tanh(|\mathbf{k}|d)\, e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \mathbf{k} \in \Lambda^*, \tag{2.2}$$

which makes the linearized problem fully explicit and is the key formula used in the computation of the dispersion relation in Chapter 3.

Third, for the higher-order analysis in Chapter 4, we require the first-order variation of $G(\eta)$ in η at the flat surface. This is given by

$$D_\eta G(0)[\dot{\eta}]\dot{\Phi} = -G_0(\dot{\eta}\, G_0\dot{\Phi}) - \nabla \cdot (\dot{\eta}\, \nabla\dot{\Phi}); \tag{2.3}$$

see [12, Lemma 1.1] or [11, Section 3.2]. This formula is used in Chapter 4 when computing the third-order Fréchet derivative of $\mathcal{F}$ and verifying the nondegeneracy of the pitchfork bifurcation.

Chapter 3

PROOF OF THEOREM 1

Following the Crandall–Rabinowitz Theorem 5, we prove the result in Theorem 1 step-by-step using lemmas.

Lemma 6 (Smoothness and mapping properties)**.** *Let $s > 2$ and define*

$$X = H^s_{\text{ee}}(\Lambda) \times H^s_{\text{oe}}(\Lambda), \quad Y = H^{s-1}_{\text{oe}}(\Lambda) \times H^{s-2}_{\text{ee}}(\Lambda).$$

Then the operator $\mathcal{F} : X \times \mathbb{R} \to Y$ is well-defined and smooth in a neighborhood of $(0, 0, c)$.

Proof. We show that $\mathcal{F}$ maps $X \times \mathbb{R}$ into Y and is smooth near the trivial solution by verifying the parity of each term in $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)^T$.

For $\mathcal{F}_1 = -c\partial_x \eta + G(\eta)\Phi$, since $\eta \in H^s_{\text{ee}}$, its x-derivative $\partial_x \eta$ is odd in x and even in y, hence $\partial_x \eta \in H^{s-1}_{\text{oe}}$. For the Dirichlet–Neumann term, the operator $G(\eta)$ depends on η through even operations and therefore preserves parity class; since $\Phi \in H^s_{\text{oe}}$, we have $G(\eta)\Phi \in H^{s-1}_{\text{oe}}$. Thus $\mathcal{F}_1 \in H^{s-1}_{\text{oe}}(\Lambda)$.

For

$$\mathcal{F}_2 = -c\partial_x \Phi + \frac{1}{2}|\nabla \Phi|^2 - \frac{(\nabla \eta \cdot \nabla \Phi + G(\eta)\Phi)^2}{2(1 + |\nabla \eta|^2)} + g\eta - \sigma H(\eta),$$

we verify that each term is even-even. Since $\Phi \in H^s_{\text{oe}}$, its x-derivative $\partial_x \Phi$ is even-even, hence $c\partial_x \Phi \in H^{s-1}_{\text{ee}}$. The term $g\eta$ is even-even since $\eta \in H^s_{\text{ee}}$ by assumption. For the curvature term $\sigma H(\eta) = \sigma \nabla \cdot (\nabla \eta / \sqrt{1 + |\nabla \eta|^2})$, since $\eta \in H^s_{\text{ee}}$, we have $\partial_x \eta \in H^s_{\text{oe}}$ and $\partial_y \eta \in H^s_{\text{eo}}$. The quantity $|\nabla \eta|^2 = (\partial_x \eta)^2 + (\partial_y \eta)^2$ is a sum of squares, hence even-even, so $1/\sqrt{1 + |\nabla \eta|^2}$ is even-even. The vector field $\nabla \eta / \sqrt{1 + |\nabla \eta|^2}$ therefore has first component $(\text{oe}) \cdot (\text{ee}) = \text{oe}$ and second component $(\text{eo}) \cdot (\text{ee}) = \text{eo}$. Taking the divergence, ∂_x of an oe function is ee and ∂_y of an eo function is ee, so $\sigma H(\eta) \in H^{s-2}_{\text{ee}}$. For $\frac{1}{2}|\nabla \Phi|^2 = \frac{1}{2}((\partial_x \Phi)^2 + (\partial_y \Phi)^2)$, the term $\partial_x \Phi$ is even-even so $(\partial_x \Phi)^2$ is even-even, and $\partial_y \Phi$ is odd in x and odd in y so $(\partial_y \Phi)^2$ is also even-even, giving $|\nabla\Phi|^2 \in H^{s-2}_{\mathrm{ee}}$. For the final nonlinear term, the inner product $\nabla\eta \cdot \nabla\Phi = \partial_x\eta\,\partial_x\Phi + \partial_y\eta\,\partial_y\Phi$ is a sum of (oe) $\cdot$ (ee) = oe and (eo) $\cdot$ (oo) = oe terms, hence odd-even. Since $G(\eta)\Phi \in H^{s-1}_{\mathrm{oe}}$, the numerator base $\nabla\eta \cdot \nabla\Phi + G(\eta)\Phi$ is odd-even, and squaring gives an even-even numerator. The denominator $1 + |\nabla\eta|^2$ is even-even, so the quotient lies in H^{s-2}_{ee}. Thus $\mathcal{F}_2 \in H^{s-2}_{\mathrm{ee}}(\Lambda)$, and $\mathcal{F} : X \times \mathbb{R} \to Y$ is well-defined.

The smoothness of $\mathcal{F}$ near the trivial solution follows from the analyticity of $\eta \mapsto G(\eta)$ as a map on Sobolev spaces, as discussed in Section 2.3. All remaining terms in $\mathcal{F}$ are polynomial or smooth nonlinear functions of η, Φ, and their derivatives, and therefore smooth as maps between the relevant Sobolev spaces for $s > 2$. $\square$

Lemma 7 (Linearization and dispersion relation)**.**

(i) *At the trivial solution, we have*

$$L_c \begin{pmatrix} \dot{\eta} \\ \dot{\Phi} \end{pmatrix} := D_{(\eta,\Phi)}\mathcal{F}(0,0;c) \begin{pmatrix} \dot{\eta} \\ \dot{\Phi} \end{pmatrix} = \begin{pmatrix} -c\partial_x & G_0 \\ g - \sigma\Delta & -c\partial_x \end{pmatrix} \begin{pmatrix} \dot{\eta} \\ \dot{\Phi} \end{pmatrix}.$$

(ii) *For a single mode* $\mathbf{k} = (\kappa_1, \kappa_2) \in \Lambda^*$, *substituting the ansatz*

$$\dot{\eta} = \hat{\eta}_{\mathbf{k}} \cos(\kappa_1 x)\cos(\kappa_2 y), \qquad \dot{\Phi} = \hat{\Phi}_{\mathbf{k}} \sin(\kappa_1 x)\cos(\kappa_2 y),$$

into $L_c(\dot{\eta}, \dot{\Phi}) = 0$ *and factoring the trigonometric terms yields the finite-dimensional system*

$$M_{\mathbf{k}}(c) \begin{pmatrix} \hat{\eta}_{\mathbf{k}} \\ \hat{\Phi}_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$M_{\mathbf{k}}(c) = \begin{pmatrix} -c\kappa_1 & |\mathbf{k}|\tanh(|\mathbf{k}|d) \\ -(g + \sigma|\mathbf{k}|^2) & c\kappa_1 \end{pmatrix}.$$

In this sense, L_c *is represented by* $M_{\mathbf{k}}(c)$ *at each Fourier mode* $\mathbf{k}$*, with* η*-modes expanding in* $\cos(\kappa_1 x)\cos(\kappa_2 y)$ *and* Φ*-modes in* $\sin(\kappa_1 x)\cos(\kappa_2 y)$*. Consequently,* L_c *acts diagonally on Fourier modes, it is completely determined by its action on each mode separately.*

(iii) *For a fixed* c*, a nontrivial solution at mode* $\mathbf{k}$ *exists if and only if* $M_{\mathbf{k}}(c)$ *is singular, which* *holds if and only if the dispersion relation*

$$D_0(\mathbf{c},\mathbf{k}) = g + \sigma|\mathbf{k}|^2 - \frac{(\mathbf{c}\cdot\mathbf{k})^2}{|\mathbf{k}|}\coth(|\mathbf{k}|d) = 0$$

is satisfied. Consequently,

$$\dim \mathscr{N}(L_c) = \#\{\mathbf{k} \in \Lambda^* : D_0(\mathbf{c},\mathbf{k}) = 0\}.$$

Proof. (i) We compute $L_c := D_{(\eta,\Phi)}\mathcal{F}|_{(0,0;\,c)}$ by linearizing each component of $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)^T$ at the trivial solution. In $\mathcal{F}_1$, the term $-c\partial_x\eta$ is already linear. For the Dirichlet–Neumann term, the linearization (2.1) gives $G_0\dot{\Phi}$, so the linearized first component is $-c\partial_x\dot{\eta} + G_0\dot{\Phi}$.

In $\mathcal{F}_2$, the terms $-c\partial_x\Phi$ and $g\eta$ are linear. The terms $|\nabla\Phi|^2$ and the nonlinear fraction are $O(\varepsilon^2)$ and therefore vanish at the trivial solution. For the curvature term, a direct computation gives

$$D_\eta[\sigma H(\eta)]\big|_{\eta=0}\dot{\eta} = \sigma\nabla\cdot(\nabla\dot{\eta}) = \sigma\Delta\dot{\eta}.$$

Combining the linearized contributions from both components, we obtain

$$L_c\begin{pmatrix}\dot{\eta}\\ \dot{\Phi}\end{pmatrix} = \begin{pmatrix}-c\partial_x\dot{\eta} + G_0\dot{\Phi}\\ -c\partial_x\dot{\Phi} + g\dot{\eta} - \sigma\Delta\dot{\eta}\end{pmatrix}, \tag{3.1}$$

which may be written in operator form as

$$L_c = \begin{pmatrix}-c\partial_x & G_0\\ g-\sigma\Delta & -c\partial_x\end{pmatrix}.$$

(ii) We seek solutions $L_c(\dot{\eta},\dot{\Phi})^T = 0$ in the symmetric subspace. Since $\dot{\eta} \in H^s_{\mathrm{ee}}$ and $\dot{\Phi} \in H^s_{\mathrm{oe}}$, we expand them in their respective Fourier bases,

$$\dot{\eta} = \sum_{\mathbf{k}'\in\Lambda^*}\hat{\eta}_{\mathbf{k}'}\cos(\kappa_1'x)\cos(\kappa_2'y), \qquad \dot{\Phi} = \sum_{\mathbf{k}'\in\Lambda^*}\hat{\Phi}_{\mathbf{k}'}\sin(\kappa_1'x)\cos(\kappa_2'y).$$

Since all differential operators and G_0 in (3.1) are Fourier multipliers — that is, each acts on a mode $e^{i\mathbf{k}\cdot\mathbf{x}}$ by multiplication by a scalar depending only on $\mathbf{k}$ — the operator L_c acts diagonally on Fourier modes. It therefore suffices to consider a single mode $\mathbf{k} = (\kappa_1, \kappa_2) \in \Lambda^*$ and substitute the ansatz

$$\dot{\eta} = \hat{\eta}_{\mathbf{k}} \cos(\kappa_1 x) \cos(\kappa_2 y), \qquad \dot{\Phi} = \hat{\Phi}_{\mathbf{k}} \sin(\kappa_1 x) \cos(\kappa_2 y).$$

Using the identities

$$\begin{aligned}
\partial_x[\cos(\kappa_1 x) \cos(\kappa_2 y)] &= -\kappa_1 \sin(\kappa_1 x) \cos(\kappa_2 y), \\
\partial_x[\sin(\kappa_1 x) \cos(\kappa_2 y)] &= \kappa_1 \cos(\kappa_1 x) \cos(\kappa_2 y), \\
\Delta[\cos(\kappa_1 x) \cos(\kappa_2 y)] &= -|\mathbf{k}|^2 \cos(\kappa_1 x) \cos(\kappa_2 y), \\
G_0[\sin(\kappa_1 x) \cos(\kappa_2 y)] &= |\mathbf{k}| \tanh(|\mathbf{k}|d) \sin(\kappa_1 x) \cos(\kappa_2 y),
\end{aligned}$$

and substituting into (3.1), we can write

$$L_c \begin{pmatrix} \dot{\eta} \\ \dot{\Phi} \end{pmatrix} = \begin{pmatrix} -c\kappa_1 & |\mathbf{k}| \tanh(|\mathbf{k}|d) \\ -(g + \sigma|\mathbf{k}|^2) & c\kappa_1 \end{pmatrix} \begin{pmatrix} \hat{\eta}_{\mathbf{k}} \sin(\kappa_1 x) \cos(\kappa_2 y) \\ \hat{\Phi}_{\mathbf{k}} \cos(\kappa_1 x) \cos(\kappa_2 y) \end{pmatrix}$$

The resulting conditions on the Fourier coefficients $(\hat{\eta}_{\mathbf{k}}, \hat{\Phi}_{\mathbf{k}})^T$ take the matrix form

$$M_{\mathbf{k}}(c) \begin{pmatrix} \hat{\eta}_{\mathbf{k}} \\ \hat{\Phi}_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

confirming that L_c is represented by $M_{\mathbf{k}}(c)$ at each Fourier mode. Note that the two components produce different trigonometric factors: sine for the first row and cosine for the second. This is consistent with the parity structure of the target spaces H^{s-1}_{oe} and H^{s-2}_{ee} respectively.

(iii) The system $M_{\mathbf{k}}(c)\mathbf{v} = 0$ has a nontrivial solution if and only if $M_{\mathbf{k}}(c)$ is singular, i.e., $\det M_{\mathbf{k}}(c) = 0$. Evaluating the determinant gives

$$-c^2\kappa_1^2 + (g + \sigma|\mathbf{k}|^2)|\mathbf{k}| \tanh(|\mathbf{k}|d) = 0.$$

Dividing through by $|\mathbf{k}| \tanh(|\mathbf{k}|d)$ yields the dispersion relation

$$D_0(\mathbf{c}, \mathbf{k}) := g + \sigma|\mathbf{k}|^2 - \frac{(\mathbf{c} \cdot \mathbf{k})^2}{|\mathbf{k}|} \coth(|\mathbf{k}|d) = 0, \quad \mathbf{k} \neq 0. \tag{3.2}$$

This relation determines which wave vectors $\mathbf{k} \in \Lambda^*$ can sustain nontrivial solutions at a given speed $\mathbf{c}$. Since L_c acts diagonally on Fourier modes, a mode $\mathbf{k}$ contributes to $\mathscr{N}(L_c)$ if and only if $D_0(c, \mathbf{k}) = 0$, and the dimension of the kernel is therefore counted by the number of such modes. When $D_0(\mathbf{c}, \mathbf{k}) = 0$, the first row of $M_{\mathbf{k}}(c)$ determines $\hat{\Phi}_{\mathbf{k}}$ uniquely in terms of $\hat{\eta}_{\mathbf{k}}$:

$$\hat{\Phi}_{\mathbf{k}} = \frac{c\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\hat{\eta}_{\mathbf{k}}.$$

Solving (3.2) for c gives

$$c^2(\mathbf{k}) = \frac{(g + \sigma|\mathbf{k}|^2)|\mathbf{k}|\tanh(|\mathbf{k}|d)}{\kappa_1^2}. \tag{3.3}$$

□

Lemma 8 (Kernel)**.** *Let* $c^* = c_p(\mathbf{k}_1) = c_p(\mathbf{k}_2)$. *If* $(\lambda, \theta) \notin \mathscr{M}_d$, *then the only solutions of the dispersion relation in* Λ^* *are* $\{\pm\mathbf{k}_1, \pm\mathbf{k}_2\}$. *Under the symmetry constraints,*

$$\mathscr{N}(L_c) = \mathrm{span}\{\zeta^*\},$$

where

$$\zeta^* = \begin{pmatrix} \cos(\kappa_1 x)\cos(\kappa_2 y) \\ \dfrac{c^*\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos(\kappa_2 y) \end{pmatrix}. \tag{3.4}$$

In particular, $\dim \mathscr{N}(L_c) = 1$.

Proof. Consider the case in which we are at the critical speed c^*.

Since L_{c^*} acts diagonally on Fourier modes, $(\dot{\eta}, \dot{\Phi}) \in \mathscr{N}(L_c)$ requires $M_{\mathbf{k}}(c^*)(\hat{\eta}_{\mathbf{k}}, \hat{\Phi}_{\mathbf{k}})^T = 0$ for every $\mathbf{k} \in \Lambda^*$. When $D_0(c^*, \mathbf{k}) = 0$ is not satisfied, the matrix $M_{\mathbf{k}}(c^*)$ is invertible, and the only solution is the trivial one.

When the dispersion relation is satisfied, i.e., $D_0(c^*, \mathbf{k}^*) = 0$, we have that $\det(M_{\mathbf{k}^*}(c^*)) = 0$. Then the rank of $M_{\mathbf{k}^*}(c^*) \leq 1$. However, note that the entry $M_{1,2}$ is $|\mathbf{k}^*|\tanh(|\mathbf{k}^*|d) > 0$. So, $M_{\mathbf{k}^*}(c^*) \neq 0$. Thus, $\mathrm{rank}(M_{\mathbf{k}^*}(c^*)) = 1$. We can use the rank-nullity theorem to show that $\dim\ker(M_{\mathbf{k}^*}(c^*)) = 1$. To find a kernel generator, we solve

$$-c^*\kappa_1\hat{\eta}_{\mathbf{k}^*} + |\mathbf{k}^*|\tanh(|\mathbf{k}^*|d)\hat{\Phi}_{\mathbf{k}^*} = 0.$$

We can uniquely determine $\hat{\Phi}_{\mathbf{k}^*}$ in terms of $\hat{\eta}_{\mathbf{k}^*}$:

$$\hat{\Phi}_{\mathbf{k}^*} = \frac{c^*\kappa_1}{|\mathbf{k}^*|\tanh(|\mathbf{k}^*|d)}\hat{\eta}_{\mathbf{k}^*}. \tag{3.5}$$

The kernel of L_{c^*} on the full space is four-dimensional since $c^* = c_p(\mathbf{k}_1) = c_p(\mathbf{k}_2)$. This would be spanned by the modes corresponding to $\pm\mathbf{k}_1, \pm\mathbf{k}_2$. Here T_1 and T_2 denote the reflections $T_1 : x \mapsto -x$ and $T_2 : y \mapsto -y$ acting on the horizontal coordinates, and ST_1 denotes the condition of restricting to functions that are symmetric (i.e., even) under T_1. Invariance under ST_1 forces η to be even in x and Φ to be odd in x, while invariance under T_2 forces both to be even in y, consistent with the symmetric subspace $X = H^s_{ee}(\Lambda) \times H^s_{oe}(\Lambda)$ defined in (1.19). We can impose the symmetries ST_1 and T_2. The condition T_2 means that the solution is invariant under $y \mapsto -y \implies \eta$ is even in y. This reduces the kernel dimension to two: only (κ_1, κ_2) and $(-\kappa_1, \kappa_2)$ remain independent. In combination with ST_1, η is now even in x as well. Since $\pm\mathbf{k}_1, \pm\mathbf{k}_2$ are all equivalent, summing over all four modes, $\dot{\eta}$ is determined by a single $\hat{\eta}_{\mathbf{k}^*}$:

$$\dot{\eta} = \hat{\eta}_{\mathbf{k}^*} \cos(\kappa_1 x) \cos(\kappa_2 y).$$

For $\dot{\Phi}$, a similar process yields $\hat{\Phi}_{\mathbf{k}^*} \sin(\kappa_1 x) \cos(\kappa_2 y)$. Note that $\dot{\Phi}$ is odd in x.

We are left with two free parameters, $\hat{\eta}_{\mathbf{k}^*}$ and $\hat{\Phi}_{\mathbf{k}^*}$. However, (3.5) shows us that $\hat{\Phi}_{\mathbf{k}}$ can be determined uniquely by $\hat{\eta}_{\mathbf{k}^*}$. Normalize by setting $\hat{\eta}_{\mathbf{k}^*} = 1$. We then have the kernel generator defined in the lemma. The kernel of L_{c^*} is then $\operatorname{span}\{\zeta^*\}$. □

Remark 9. *By assumption $(\lambda, \theta) \notin \mathscr{M}_d$, which by equivalence is the dispersion relation having no solutions in Λ^* other than $\pm\mathbf{k}_1, \pm\mathbf{k}_2$. The matrix $M_{\mathbf{k}}(c^*)$ is invertible for all other modes, forcing a trivial solution. Thus, the kernel of L_{c^*} on the full space is four-dimensional. Refer to Appendix A.*

Lemma 10 (Fredholm property and range characterization)**.** *The operator $L_{c^*} : X \to Y$ is Fredholm of index zero. We will show this by proving*

$$\dim \mathscr{N}(L_{c^*}) = 1 \quad \textit{and} \quad \operatorname{codim} \mathscr{R}(L_{c^*}) = 1.$$

Moreover,

$$(f_1, f_2)^T \in \mathscr{R}(L_{c^*}) \textit{ if and only if } \langle (f_1, f_2)^T, \mathbf{w}^* \rangle = 0,$$

where

$$\mathbf{w}^* := \begin{pmatrix} (g + \sigma|\mathbf{k}|^2) \sin(\kappa_1 x) \cos(\kappa_2 y) \\ -c^* \kappa_1 \cos(\kappa_1 x) \cos(\kappa_2 y) \end{pmatrix}. \tag{3.6}$$

Proof. For all non-critical modes, the matrix $M_{\mathbf{k}}(c^*)$ is invertible. We are concerned with the case of the critical mode, where the determinant is 0. To characterize, we will use the finite-dimensional

Fredholm Alternative: for a linear map A, the equation $A\mathbf{v} = \mathbf{f}$ has a solution if and only if $\mathbf{f}$ is orthogonal to every vector of $\mathscr{N}(A^T)$. We can write this condition as

$$\mathbf{f} \in \mathscr{R}(A) \text{ if and only if } \langle \mathbf{f}, \mathbf{w}^* \rangle = 0 \text{ for all } \mathbf{w}^* \in \mathscr{N}(A^T). \tag{3.7}$$

We can apply this theorem to our case since the equality is finite-dimensional at each Fourier mode. At each mode, we have a 2×2 linear system. Recall, we are working with $\mathbf{k} = \mathbf{k}_1$ Applying the finite-dimensional Fredholm Alternative (3.7) to the matrix system at the critical mode $\mathbf{k} = \mathbf{k}_1$, the system

$$M_{\mathbf{k}}(c^*) \begin{pmatrix} \hat{\eta}_{\mathbf{k}} \\ \hat{\Phi}_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \hat{f}_{1,\mathbf{k}} \\ \hat{f}_{2,\mathbf{k}} \end{pmatrix}$$

is solvable if and only if $(\hat{f}_{1,\mathbf{k}}, \hat{f}_{2,\mathbf{k}})^T \perp \mathscr{N}(M_{\mathbf{k}}(c^*)^T)$. We therefore compute the transpose as

$$M_{\mathbf{k}}(c^*)^T = \begin{pmatrix} -c^*\kappa_1 & -(g + \sigma|\mathbf{k}|^2) \\ |\mathbf{k}| \tanh(|\mathbf{k}|d) & c^*\kappa_1 \end{pmatrix},$$

and determine its kernel by solving $M_{\mathbf{k}}(c^*)^T \mathbf{w}^* = 0$. Setting $\mathbf{w}^* = (w_1, w_2)^T$, the first row gives

$$-c^*\kappa_1 w_1 - (g + \sigma|\mathbf{k}|^2) w_2 = 0.$$

Choosing $w_2 = -c^*\kappa_1$ yields $w_1 = g + \sigma|\mathbf{k}|^2$, and since $M_{\mathbf{k}}(c^*)^T$ has rank one at the critical mode, the kernel is exactly one-dimensional,

$$\mathscr{N}\big(M_{\mathbf{k}}(c^*)^T\big) = \operatorname{span} \left\{ \begin{pmatrix} g + \sigma|\mathbf{k}|^2 \\ -c^*\kappa_1 \end{pmatrix} \right\}.$$

We now want to write this as a condition on the functions f_1, f_2 rather than their Fourier coefficients. The condition $(\hat{f}_{1,\mathbf{k}}, \hat{f}_{2,\mathbf{k}})^T \perp \mathscr{N}(M_{\mathbf{k}}(c^*))^T$ gives us:

$$(g + \sigma|\mathbf{k}|^2)\hat{f}_{1,\mathbf{k}} - c^*\kappa_1 \hat{f}_{2,\mathbf{k}} = 0. \tag{3.8}$$

Since $\mathbf{f} := (f_1, f_2)^T \in Y$, we have that $f_1 \in H^{s-1}_{\mathrm{oe}}(\Lambda)$ and $f_2 \in H^{s-2}_{\mathrm{ee}}(\Lambda)$. So, f_1 expands in the basis

$\{\sin(\kappa_1 x)\cos(\kappa_2 y) : (\kappa_1, \kappa_2) \in \Lambda^*\}$ as

$$f_1 = \sum_{\mathbf{k}' \in \Lambda^*} \hat{f}_{1,\mathbf{k}'} \sin(\kappa_1' x)\cos(\kappa_2' y).$$

We take the L^2 inner product of both sides with $\sin(\kappa_1 x)\cos(\kappa_2 y)$. We also use orthogonality of the Fourier basis (all terms vanish except the one in our critical mode $\mathbf{k}$):

$$\langle f_1, \sin(\kappa_1 x)\cos(\kappa_2 y)\rangle_{L^2} = \hat{f}_{1,\mathbf{k}} \cdot \|\sin(\kappa_1 x)\cos(\kappa_2 y)\|_{L^2}^2.$$

Thus,

$$\hat{f}_{1,\mathbf{k}} = \frac{\langle f_1, \sin(\kappa_1 x)\cos(\kappa_2 y)\rangle}{\|\sin(\kappa_1 x)\cos(\kappa_2 y)\|_{L^2}^2}.$$

Applying the same argument for $\hat{f}_2$ gives

$$\hat{f}_{2,\mathbf{k}} = \frac{\langle f_2, \cos(\kappa_1 x)\cos(\kappa_2 y)\rangle}{\|\cos(\kappa_1 x)\cos(\kappa_2 y)\|_{L^2}^2}.$$

We use the fact that $\|\cos(\kappa_1 x)\cos(\kappa_2 y)\|_{L_2}^2 = \|\sin(\kappa_1 x)\cos(\kappa_2 y)\|_{L_2}^2$, which is true since sine and cosine have the same L^2 norm over a full period.

Substituting what we have for $\hat{f}_{1,\mathbf{k}}$ and $\hat{f}_{2,\mathbf{k}}$ in the orthogonal condition (3.8), we obtain

$$(g + \sigma|\mathbf{k}|^2)\frac{\langle f_1, \sin(\kappa_1 x)\cos(\kappa_2 y)\rangle}{\|\sin(\kappa_1 x)\cos(\kappa_2 y)\|_{L^2}^2} - c^*\kappa_1 \frac{\langle f_2, \cos(\kappa_1 x)\cos(\kappa_2 y)\rangle}{\|\sin(\kappa_1 x)\cos(\kappa_2 y)\|_{L^2}^2} = 0.$$

Multiplying with the denominator (> 0) on both sides and using linearity of the L^2 inner product:

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} (g + \sigma|\mathbf{k}|^2)\sin(\kappa_1 x)\cos(\kappa_2 y) \\ -c^*\kappa_1 \cos(\kappa_1 x)\cos(\kappa_2 y) \end{pmatrix} \right\rangle = 0.$$

This is then the desired condition $\langle \mathbf{f}, \mathbf{w}^*\rangle = 0$. Since $\mathbf{w}^*$ is spanned by a single vector, it is one-dimensional.

To show Fredholmness, it remains to show that the range of the linearized operator is closed. For this, suppose $\{f^{(n)}\} \subset \mathscr{R}(L_{c^*})$ with $f^{(n)} \to f$ in Y. Each $f^{(n)}$ satisfies the solvability condition (3.8):

$$(g + \sigma|\mathbf{k}|^2)\hat{f}_{1,\mathbf{k}}^{(n)} - c^*\kappa_1 \hat{f}_{2,\mathbf{k}}^{(n)} = 0.$$

Since convergence in Y implies convergence of Fourier coefficients, as the map $f \mapsto \hat{f}_{\mathbf{k}}$ is continuous, we can pass to the limit to obtain:

$$(g + \sigma|\mathbf{k}|^2)\hat{f}_{1,\mathbf{k}} - c^*\kappa_1\hat{f}_{2,\mathbf{k}} = 0,$$

so $f \in \mathscr{R}(L_{c^*})$. Hence, the range is closed.

Finally, for the Fredholm index, we have $\dim \mathscr{N}(L_{c^*}) = 1$ and $\operatorname{codim} \mathscr{R}(L_{c^*}) = 1$. So,

$$\operatorname{ind}(L_{c^*}) = \dim \mathscr{N}(L_{c^*}) - \operatorname{codim} \mathscr{R}(L_{c^*}) = 1 - 1 = 0.$$

Hence, L_{c^*} is Fredholm with index 0. □

Lemma 11 (Transversality)**.** *The transversality condition holds:*

$$D_cL_{c^*}[\zeta^*] \notin \mathscr{R}(L_{c^*}),$$

equivalently,

$$\langle D_cL_{c^*}[\zeta^*], \mathbf{w}^*\rangle_{L^2} \neq 0. \tag{3.9}$$

Proof. By the range characterization of Lemma 10, $f \notin \mathscr{R}(L_{c^*})$ if and only if $\langle f, \mathbf{w}^*\rangle_{L^2} \neq 0$. We therefore compute $\langle D_cL_{c^*}[\zeta^*], \mathbf{w}^*\rangle_{L^2}$ directly.

Differentiating L_c with respect to c gives

$$D_cL_c \begin{pmatrix} \dot{\eta} \\ \dot{\Phi} \end{pmatrix} = \begin{pmatrix} -\partial_x\dot{\eta} \\ -\partial_x\dot{\Phi} \end{pmatrix}.$$

Applying this to the kernel generator ζ^* from (3.4):

$$D_cL_{c^*}[\zeta^*] = \begin{pmatrix} \kappa_1 \sin(\kappa_1 x)\cos(\kappa_2 y) \\ -\dfrac{c^*\kappa_1^2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)} \cos(\kappa_1 x)\cos(\kappa_2 y) \end{pmatrix}.$$

Taking the inner product with

$$\mathbf{w}^* = \begin{pmatrix} (g + \sigma|\mathbf{k}|^2)\sin(\kappa_1 x)\cos(\kappa_2 y) \\ -c^*\kappa_1\cos(\kappa_1 x)\cos(\kappa_2 y) \end{pmatrix},$$

and using $\|\sin(\kappa_1 x)\cos(\kappa_2 y)\|_{L^2}^2 = \|\cos(\kappa_1 x)\cos(\kappa_2 y)\|_{L^2}^2 =: N > 0$:

$$\begin{aligned}
\langle D_c L_{c^*}[\zeta^*], \mathbf{w}^* \rangle_{L^2} &= \kappa_1 (g + \sigma |\mathbf{k}|^2) N + \frac{(c^*)^2 \kappa_1^2}{|\mathbf{k}| \tanh(|\mathbf{k}| d)} N \\
&= N\kappa_1 \left[(g + \sigma |\mathbf{k}|^2) + \frac{(c^*)^2 \kappa_1^2}{|\mathbf{k}| \tanh(|\mathbf{k}| d)} \right] \\
&= N\kappa_1 \left[(g + \sigma |\mathbf{k}|^2) + (g + \sigma |\mathbf{k}|^2) \right] \\
&= 2N\kappa_1 (g + \sigma |\mathbf{k}|^2),
\end{aligned}$$

where we used the dispersion relation $(c^*)^2 \kappa_1^2 = (g + \sigma |\mathbf{k}|^2)|\mathbf{k}| \tanh(|\mathbf{k}| d)$ in the third line. Since $N > 0$, $\kappa_1 > 0$, $g > 0$, and $\sigma > 0$, the inner product is strictly positive and in particular nonzero, which establishes the transversality condition. □

Chapter 4

BIFURCATION ANALYSIS

4.1 Invertibility of the linearized operator

Having established the existence of the bifurcating branch $(\eta(s), \Phi(s), c(s))$ in the preceding section, we now take a closer look at the linearized operator along this branch. For each $s \in (-\varepsilon, \varepsilon)$, define

$$L(s) := D_{(\eta,\Phi)}\mathcal{F}(\eta(s), \Phi(s), c(s)),$$

which is the Fréchet derivative of $\mathcal{F}$ evaluated at the nontrivial solution at parameter s. At $s = 0$, this reduces to the critical operator $L(0) = L_{c^*}$, which is Fredholm of index zero with a one-dimensional kernel spanned by ζ^*. For $s \neq 0$, the question is whether $L(s)$ becomes invertible, i.e., whether the kernel of the linearized operator becomes trivial as one moves off the bifurcation point along the branch. In this section, we prove the following theorem.

Theorem 12. *Let the assumptions of Theorem 1 hold, and let $(\eta(s), \Phi(s), c(s))$ denote the bifurcating branch of diamond wave solutions. Then the following hold.*

(i) *The linearized operator $L(s) := D_{(\eta,\Phi)}\mathcal{F}(\eta(s), \Phi(s), c(s))$ is invertible for all $0 < |s| \ll 1$.*

(ii) *The resulting bifurcation is a supercritical pitchfork.*

Proof. (i) To streamline notation, we write $u(s) := (\eta(s), \Phi(s)) \in X$ for the state variable along the branch.

We concluded in Lemma 10 that L_{c^*} is a Fredholm operator of index zero with a simple eigenvalue 0. The perturbation theory of Kato [13], which guarantees the continuous dependence of simple eigenvalues under smooth perturbations of the operator, ensures that for s in a neighborhood of 0, the operator $L(s)$ has a unique simple eigenvalue $\mu_0(s)$ and corresponding eigenfunction $\zeta_0(s)$, both depending smoothly on s, with $\mu_0(0) = 0$ and $\zeta_0(0) = \zeta^*$. These satisfy

$$L(s)\zeta_0(s) = D_u\mathcal{F}(u(s), c(s))\zeta_0(s) = \mu_0(s)\zeta_0(s). \tag{4.1}$$

Invertibility of $L(s)$ for $s \neq 0$ is then equivalent to showing $\mu_0(s) \neq 0$ for small $s \neq 0$.

To determine the behavior of $\mu_0(s)$, we apply the Lyapunov–Schmidt reduction. Let $X = \mathscr{N}(L_{c^*}) \oplus X_0$ and $Y = \mathscr{R}(L_{c^*}) \oplus \text{span}\{\mathbf{w}^*\}$. The equation $\mathcal{F}(u, c) = 0$ in a neighborhood of $(0, c^*)$ is equivalent to a reduced scalar bifurcation equation,

$$\tilde{\mathcal{B}}(t, c) := \langle \mathcal{F}(t\zeta^* + \tilde{\psi}(t, c), c), \mathbf{w}^* \rangle = 0,$$

where $\tilde{\psi}(t, c) \in X_0$ is the unique solution to the range equation $(I - Q)\mathcal{F}(t\zeta^* + \tilde{\psi}, c) = 0$, with Q being the L^2 projection onto $\text{span}\{\mathbf{w}^*\}$.

Since the trivial solution $\mathcal{F}(0, c) = 0$ exists for all c, we have $\tilde{\psi}(0, c) = 0$ and consequently $\tilde{\mathcal{B}}(0, c) = 0$. It follows that $t = 0$ is always a root of $\tilde{\mathcal{B}}$, and therefore $\tilde{\mathcal{B}}$ factors as [14, (I.5.9)–(I.5.10)], which establishes the factoring of the reduced equation through the trivial root,

$$\tilde{\mathcal{B}}(t, c) = t\hat{\mathcal{B}}(t, c).$$

The nontrivial bifurcating branch $(u(s), c(s))$ corresponds to the roots of the factored equation $\hat{\mathcal{B}}(t, c(t)) = 0$. Differentiating once at $t = 0$ and solving for $\dot{c}(0)$ gives

$$\dot{c}(0) = -\frac{\partial_t\hat{\mathcal{B}}(0, c^*)}{\partial_c\hat{\mathcal{B}}(0, c^*)} = -\frac{\frac{1}{2}\langle D^2_{uu}\mathcal{F}(0, c^*)[\zeta^*, \zeta^*], \mathbf{w}^*\rangle}{\langle D^2_{cu}\mathcal{F}(0, c^*)[\zeta^*], \mathbf{w}^*\rangle},$$

where the numerator is computed via [14, (I.6.2)], which gives the formula for the first bifurcation derivative in terms of second-order Fréchet derivatives. The numerator vanishes: by (5.6), the term $D^2_{uu}\mathcal{F}(0, c^*)[\zeta^*, \zeta^*]$ has the wrong parity to have a nonzero inner product with $\mathbf{w}^*$, since $\mathcal{F}$ maps even functions to even functions and odd to odd. The denominator is the transversality condition (3.9), which is nonzero by Lemma 11. Hence $\dot{c}(0) = 0$.

Since $\dot{c}(0) = 0$, the relation [14, (I.7.40)], which links the derivative of the eigenvalue to the derivative of the bifurcation parameter, gives $\dot{\mu}_0(0) = 0$, and we must go to higher order. Differentiating $\hat{\mathcal{B}}(t, c(t)) = 0$ twice at $t = 0$ and using $\dot{c}(0) = 0$ gives

$$\partial^2_{tt}\hat{\mathcal{B}}(0, c^*) + \partial_c\hat{\mathcal{B}}(0, c^*)\ddot{c}(0) = 0,$$

which yields, by the second bifurcation formula [14, (I.6.4)–(I.6.11)],

$$\ddot{c}(0) = -\frac{1}{3}\frac{\langle D^3_{uuu}\mathcal{F}(0,c^*)[\zeta^*,\zeta^*,\zeta^*],\mathbf{w}^*\rangle}{\langle D^2_{cu}\mathcal{F}(0,c^*)[\zeta^*],\mathbf{w}^*\rangle}. \tag{4.2}$$

The denominator is nonzero by the transversality condition (3.9), and the numerator is nonzero by Appendix B, where it is computed explicitly. Hence $\ddot{c}(0) \neq 0$.

It then follows from the formula in [14, (I.7.45)], which relates the second derivative of the critical eigenvalue to the second derivative of the wave speed along the branch, that

$$2\langle D_c L_{c^*}, \mathbf{w}^*\rangle\, \ddot{c}(0) = -\ddot{\mu}_0(0).$$

Since the left-hand side is nonzero by the transversality condition (3.9) and $\ddot{c}(0) \neq 0$, we conclude $\ddot{\mu}_0(0) \neq 0$.

Finally since $\mu_0(0) = 0$, $\dot{\mu}_0(0) = 0$, and $\ddot{\mu}_0(0) \neq 0$, the Taylor expansion gives

$$\mu_0(s) = \tfrac{1}{2}\ddot{\mu}_0(0)\, s^2 + O(s^3) \neq 0$$

for small $s \neq 0$. Therefore, $L(s)$ is invertible for $0 < |s| \ll 1$.

(ii) The type of bifurcation follows from the sign of $\ddot{c}(0)$, which is determined in Appendix B. By [14, (I.6.11)], which classifies the pitchfork as sub- or supercritical according to the sign of $\ddot{c}(0)$, the positivity of $\ddot{c}(0)$ confirms that the bifurcation is supercritical. □

The corresponding bifurcation diagram is depicted in Figure 4.1: the nontrivial branch opens to the right of c^*, and the trivial flat-water state loses stability as c increases through c^*.

The supercritical nature of the bifurcation has a clear physical interpretation. The diamond wave solutions emerge from the flat-water state as the wave speed c increases through the critical value c^*, and they exist for wave speeds slightly above c^*. In other words, the nonlinear wave pattern is sustained at speeds faster than the critical speed predicted by the linear dispersion relation. This is in contrast to a subcritical bifurcation, where nontrivial solutions would exist at speeds below c^* and the bifurcating branch would bend back toward smaller wave speeds.

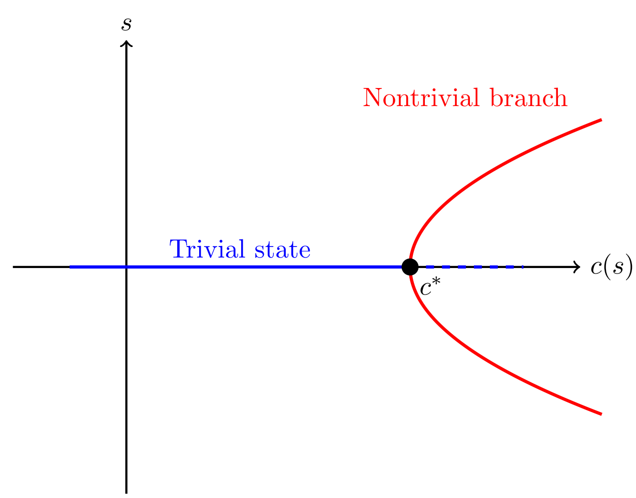

Figure 4.1: Supercritical pitchfork bifurcation diagram. The solid blue line is the trivial flat-water state, the dashed blue line indicates its instability for $c > c^*$, and the red curve is the bifurcating branch of nontrivial diamond wave solutions.

Chapter 5

APPENDIX

A Equivalence of the Dispersion Relation and the Reeder–Shinbrot Formulation

We want to show that a mode $\mathbf{k}' \in \Lambda^*$ contributes nontrivially to the kernel $\ker L_{c^*}$ if and only if it satisfies the dispersion relation

$$\mathcal{D}_0(c^*, \mathbf{k}') = 0.$$

Consequently, the dimension of $\ker L_{c^*}$ is equal to the number of solutions of this relation in Λ^*. We show that, under the condition $(\lambda, \theta) \notin \mathscr{M}_d$, the only solutions are the fundamental modes

$$\{\pm\mathbf{k}_1, \pm\mathbf{k}_2\},$$

and that this condition is equivalent to the formulation of Reeder–Shinbrot. Every $\mathbf{k}' \in \Lambda^*$ can be written as

$$\mathbf{k}' = (p\kappa_1, q\kappa_2), \quad p, q \in \mathbb{Z},$$

with magnitude

$$|\mathbf{k}'| = \sqrt{p^2\kappa_1^2 + q^2\kappa_2^2}.$$

The fundamental diamond-wave modes correspond to $(p, q) \in \{(\pm1, \pm1)\}$. Then the condition $\mathcal{D}_0(c^*, \mathbf{k}') = 0$ becomes

$$g + \sigma(p^2\kappa_1^2 + q^2\kappa_2^2) = \frac{(c^*)^2 p^2 \kappa_1^2}{\sqrt{p^2\kappa_1^2 + q^2\kappa_2^2}} \coth\left(\sqrt{p^2\kappa_1^2 + q^2\kappa_2^2}\, d\right). \tag{5.1}$$

The critical speed c^* is determined from the fundamental mode $(p, q) = (1, 1)$:

$$(c^*)^2 = \frac{(g + \sigma|\mathbf{k}_1|^2)|\mathbf{k}_1| \tanh(|\mathbf{k}_1|d)}{\kappa_1^2}. \tag{5.2}$$

Write

$$\kappa_1 = |\mathbf{k}_1| \cos\theta, \quad \kappa_2 = |\mathbf{k}_1| \sin\theta,$$

and define

$$\rho_{pq}^2 := p^2 \cos^2\theta + q^2 \sin^2\theta.$$

Then

$$|\mathbf{k}'| = |\mathbf{k}_1|\rho_{pq}.$$

Introduce the dimensionless parameter

$$\lambda := \frac{1}{|\mathbf{k}_1|}\sqrt{\frac{g}{\sigma}}, \quad d_* := |\mathbf{k}_1|d.$$

Substituting (5.2) into (5.1) and simplifying yields

$$(\lambda^2 + \rho_{pq}^2)\rho_{pq} \tanh(\rho_{pq}d_*) = p^2(\lambda^2 + 1) \tanh(d_*). \tag{5.3}$$

Setting

$$p = m + n, \quad q = m - n,$$

we obtain

$$\rho_{mn}^2 = (m + n)^2 \cos^2\theta + (m - n)^2 \sin^2\theta,$$

and (5.3) becomes

$$(\lambda^2 + \rho_{mn}^2)\rho_{mn} \tanh(\rho_{mn}d_*) = (\lambda^2 + 1)(m + n)^2 \tanh(d_*), \tag{5.4}$$

which is precisely the condition appearing in Reeder–Shinbrot.

Thus, the dispersion relation admits a non-fundamental solution $\mathbf{k}' \notin \{\pm\mathbf{k}_1, \pm\mathbf{k}_2\}$ if and only if (5.4) holds for some $(m, n) \neq (1, 0)$.

The forbidden set $\mathscr{M}_d$ is exactly the set of parameters (λ, θ) for which such additional solutions exist. Therefore, when $(\lambda, \theta) \notin \mathscr{M}_d$, the only solutions of the dispersion relation are the four fundamental modes, and

$$\dim \mathscr{N}(L_{c^*}) = 4$$

on the full space (prior to symmetry reduction).

B Verification of the pitchfork nondegeneracy condition

Recall our main operator $\mathcal{F}$, kernel generator ζ^*, and adjoint kernel generator $\mathbf{w}^*$:

$$\mathcal{F}(\eta, \Phi; c) := \begin{pmatrix} -c\,\partial_x \eta + G(\eta)\Phi \\ -c\,\partial_x \Phi + \dfrac{1}{2}|\nabla\Phi|^2 - \dfrac{\big(\nabla\eta\cdot\nabla\Phi + G(\eta)\Phi\big)^2}{2(1+|\nabla\eta|^2)} + g\eta - \sigma H(\eta) \end{pmatrix}, \tag{1.25}$$

$$\zeta^* = \begin{pmatrix} \cos(\kappa_1 x)\cos(\kappa_2 y) \\ \dfrac{c^*\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos(\kappa_2 y) \end{pmatrix}, \tag{3.4}$$

$$\mathbf{w}^* := \begin{pmatrix} (g+\sigma|\mathbf{k}|^2)\sin(\kappa_1 x)\cos(\kappa_2 y) \\ -c^*\kappa_1\cos(\kappa_1 x)\cos(\kappa_2 y) \end{pmatrix}. \tag{3.6}$$

Examining the terms within our operator $\mathcal{F}$, note that only some terms have a cubic contribution. In $\mathcal{F}_1$, only the Dirichlet–Neumann term $G(\eta)\Phi$ contributes, and in $\mathcal{F}_2$, the curvature and nonlinear fraction contribute.

Cubic contributions

We compute the cubic contributions to $D^3_{uuu}\mathcal{F}[\zeta^*, \zeta^*, \zeta^*]$ from each component of $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)^T$ separately.

For $\mathcal{F}_1$, the only cubic contribution comes from the Dirichlet–Neumann term, since $-c\partial_x\eta$ is linear and therefore vanishes under the third variation. Using the Taylor expansion of $G(\eta)$, the third variation picks out the term with two variations on G and one on Φ, giving

$$D^3_{uuu}\mathcal{F}_1[\zeta^*, \zeta^*, \zeta^*] = 3D^2_\eta G(0)[\dot\eta, \dot\eta]\dot\Phi. \tag{5.5}$$

For $\mathcal{F}_2$, we set $(\eta, \Phi) = t(\dot\eta, \dot\Phi)$, expand each term in powers of t, and extract the t^3 coefficient. We treat each term in turn.

For the curvature term $\sigma H(t\dot\eta)$, we expand

$$H(t\dot\eta) = \nabla \cdot \frac{t\nabla\dot\eta}{\sqrt{1+t^2|\nabla\dot\eta|^2}},$$

and use the Taylor expansion $(1+t^2|\nabla\dot\eta|^2)^{-1/2} = 1 - \frac{t^2}{2}|\nabla\dot\eta|^2 + O(t^4)$ to obtain

$$H(t\dot\eta) = t\Delta\dot\eta - \frac{t^3}{2}\nabla\cdot(|\nabla\dot\eta|^2\nabla\dot\eta) + O(t^5).$$

The cubic coefficient is therefore

$$\sigma H(t\dot\eta) = -\frac{\sigma}{2}\nabla\cdot(|\nabla\dot\eta|^2\nabla\dot\eta).$$

For the last nonlinear fraction, we define the numerator $N(t) := \nabla(t\dot\eta)\cdot\nabla(t\dot\Phi) + G(t\dot\eta)(t\dot\Phi)$ and expand each part. The gradient term gives $\nabla(t\dot\eta)\cdot\nabla(t\dot\Phi) = t^2\nabla\dot\eta\cdot\nabla\dot\Phi$, and the Dirichlet–Neumann term expands as

$$G(t\dot\eta)(t\dot\Phi) = tG_0\dot\Phi + t^2 D_\eta G(0)[\dot\eta]\dot\Phi + O(t^3),$$

so that

$$N(t) = tG_0\dot\Phi + t^2\left(\nabla\dot\eta\cdot\nabla\dot\Phi + D_\eta G(0)[\dot\eta]\dot\Phi\right) + O(t^3).$$

Squaring and expanding the denominator $2(1+|\nabla(t\dot\eta)|^2)^{-1} = 1 - t^2|\nabla\dot\eta|^2 + O(t^4)$, we find

$$\frac{N(t)^2}{2(1+|\nabla(t\dot\eta)|^2)} = \frac{t^2(G_0\dot\Phi)^2}{2} + t^3 G_0\dot\Phi\left(\nabla\dot\eta\cdot\nabla\dot\Phi + D_\eta G(0)[\dot\eta]\dot\Phi\right) + O(t^4).$$

The cubic coefficient is therefore

$$\frac{N(t)^2}{2(1+|\nabla(t\dot\eta)|^2)} = G_0\dot\Phi\left(\nabla\dot\eta\cdot\nabla\dot\Phi + D_\eta G(0)[\dot\eta]\dot\Phi\right).$$

Combining all cubic contributions from $\mathcal{F}_2$, and recalling that $D^3_{uuu}\mathcal{F}_2[\zeta^*,\zeta^*,\zeta^*]$ equals 6 times the t^3 coefficient, we obtain

$$D^3_{uuu}\mathcal{F}_2[\zeta^*,\zeta^*,\zeta^*] = -3\sigma\nabla\cdot(|\nabla\dot\eta|^2\nabla\dot\eta) + 6G_0\dot\Phi\left(\nabla\dot\eta\cdot\nabla\dot\Phi + D_\eta G(0)[\dot\eta]\dot\Phi\right). \tag{5.6}$$

Finally, we note that by the first row of $L_{c^*}\zeta^* = 0$, we have the relation $G_0\dot\Phi = -c^*\partial_x\dot\eta$, so every occurrence of $G_0\dot\Phi$ is expressible in terms of $\dot\eta$ and its derivatives, which we exploit when projecting onto $\mathbf{w}^*$ in the next subsection.

First component computation

Differentiating the first-order variation of $G(\eta)$ given in (2.3), we have

$$\langle D_\eta^2 G(0)[\dot\eta,\dot\eta],\dot\Phi\rangle = -(D_\eta G_0\dot\eta)(\dot\eta\, G_0\dot\Phi) - G_0(\dot\eta\, G_0(\dot\eta\, G_0\dot\Phi)) + G_0(\dot\eta\,\nabla\cdot(\dot\eta\,\nabla\dot\Phi)) - G_0(\nabla\dot\eta\cdot\nabla\dot\Phi) + \nabla\cdot(\dot\eta\, G_0\dot\Phi)\nabla\dot\eta.$$

We eliminate $\dot\Phi$ in favor of $\dot\eta$ using the relation $G_0\dot\Phi = -c^*\partial_x\dot\eta$, so that

$$\dot\Phi = -G_0^{-1}(c^*\partial_x\dot\eta), \qquad G_0^{-1}e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\, e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{k}\in\Lambda^*.$$

Substituting into each term gives

$$\begin{aligned}
\langle D_\eta^2 G(0)[\dot\eta,\dot\eta],\dot\Phi\rangle = {}& -(D_\eta G_0\dot\eta)\big(\dot\eta\,(-c^*\partial_x\dot\eta)\big) \\
& - G_0\big(\dot\eta\, G_0(\dot\eta\,(-c^*\partial_x\dot\eta))\big) \\
& + G_0\Big(\dot\eta\,\nabla\cdot\big(\dot\eta\,\nabla(-G_0^{-1}(c^*\partial_x\dot\eta))\big)\Big) \\
& - G_0\big(\nabla\dot\eta\cdot\nabla(-G_0^{-1}(c^*\partial_x\dot\eta))\big) \\
& + \nabla\cdot(\dot\eta\,(-c^*\partial_x\dot\eta))\,\nabla\dot\eta.
\end{aligned}$$

We label these five terms T_1 through T_5 and examine each in turn.

Before computing, we observe that the projection $\langle\cdot,\mathbf{w}_1^*\rangle_{L^2}$ picks out only the Fourier coefficient at the critical mode (κ_1,κ_2), i.e., only the component proportional to $\sin(\kappa_1 x)\cos(\kappa_2 y)$. By orthogonality of the Fourier basis,

$$\int_\Lambda \sin(m\kappa_1 x)\cos(n\kappa_2 y)\cdot\sin(\kappa_1 x)\cos(\kappa_2 y)\,d\mathbf{x} = 0 \quad \text{for } (m,n)\neq(1,1).$$

Therefore, any term whose Fourier support does not include the mode (κ_1,κ_2) vanishes immediately under the projection, without any explicit computation. Furthermore, the product-to-sum identities

$$\sin(px)\sin(qx) = \tfrac12[\cos((p-q)x) - \cos((p+q)x)], \quad \sin(px)\cos(qx) = \tfrac12[\sin((p+q)x) + \sin((p-q)x)]$$

show that the product of two functions at frequencies p and q produces only frequencies $p+q$ and $|p-q|$. In particular, if both factors live at frequency $2\kappa_1$ in x, their product lives at frequencies

0 and $4\kappa_1$, and the critical frequency κ_1 is never generated. We use this to dispose of T_1 and T_4 immediately.

Term 1: $T_1 := -(D_\eta G_0 \dot\eta)\big(\dot\eta\,(-c^*\partial_x\dot\eta)\big)$**.** We first determine the Fourier support of each factor. The product $\dot\eta\,(-c^*\partial_x\dot\eta)$ is

$$\dot\eta\,(-c^*\partial_x\dot\eta) = c^*\kappa_1\cos(\kappa_1 x)\sin(\kappa_1 x)\cos^2(\kappa_2 y) = \frac{c^*\kappa_1}{4}\sin(2\kappa_1 x) + \frac{c^*\kappa_1}{4}\sin(2\kappa_1 x)\cos(2\kappa_2 y),$$

which is supported only at modes $(2\kappa_1, 0)$ and $(2\kappa_1, 2\kappa_2)$.

For the factor $D_\eta G(0)[\dot\eta]\dot\Phi$, we use the linearization formula for the Dirichlet–Neumann operator:

$$D_\eta G(0)[\dot\eta]\dot\Phi = -G_0(\dot\eta\,G_0\dot\Phi) - \nabla\cdot(\dot\eta\,\nabla\dot\Phi).$$

For the first part, $\dot\eta\,G_0\dot\Phi = \dot\eta\,(-c^*\partial_x\dot\eta)$ lives at modes $(2\kappa_1, 0)$ and $(2\kappa_1, 2\kappa_2)$, and since G_0 acts as a Fourier multiplier it preserves this support, so $G_0(\dot\eta\,G_0\dot\Phi)$ remains at those same modes. For the second part, since $\dot\eta = \cos(\kappa_1 x)\cos(\kappa_2 y)$ carries frequency κ_1 in x and $\nabla\dot\Phi$ also carries frequency κ_1 in x, their product $\dot\eta\,\nabla\dot\Phi$ lands at frequency $2\kappa_1$ in x, and hence so does $\nabla\cdot(\dot\eta\,\nabla\dot\Phi)$. Therefore $D_\eta G(0)[\dot\eta]\dot\Phi$ is entirely supported at modes $(2\kappa_1, 0)$ and $(2\kappa_1, 2\kappa_2)$.

Now T_1 is the pointwise product of two functions, each at frequency $2\kappa_1$ in x. By the product-to-sum identities:

$$\sin(2\kappa_1 x)\cdot\sin(2\kappa_1 x) = \tfrac{1}{2}(1-\cos(4\kappa_1 x)), \qquad \sin(2\kappa_1 x)\cdot\cos(2\kappa_1 x) = \tfrac{1}{2}\sin(4\kappa_1 x).$$

The product therefore lives only at frequencies 0 and $4\kappa_1$ in x. Since $\{0, 4\kappa_1\} \not\ni \kappa_1$, the critical mode (κ_1, κ_2) never appears in T_1, and by orthogonality

$$\langle T_1, \mathbf{w}_1^*\rangle_{L^2} = 0.$$

Term 2: $T_2 := -G_0\big(\dot\eta\,G_0(\dot\eta\,(-c^*\partial_x\dot\eta))\big)$**.** We have

$$\dot\eta\,(-c^*\partial_x\dot\eta) = \frac{c^*\kappa_1}{4}\sin(2\kappa_1 x) + \frac{c^*\kappa_1}{4}\sin(2\kappa_1 x)\cos(2\kappa_2 y).$$

Applying G_0, using that $\sin(2\kappa_1 x)$ has frequency magnitude $2\kappa_1$ and $\sin(2\kappa_1 x)\cos(2\kappa_2 y)$ has frequency magnitude $2|\mathbf{k}|$:

$$G_0(\dot\eta\,(-c^*\partial_x\dot\eta)) = \frac{c^*\kappa_1^2}{2}\tanh(2\kappa_1 d)\sin(2\kappa_1 x) + \frac{c^*\kappa_1|\mathbf{k}|}{2}\tanh(2|\mathbf{k}|d)\sin(2\kappa_1 x)\cos(2\kappa_2 y).$$

Multiplying by $\dot\eta = \cos(\kappa_1 x)\cos(\kappa_2 y)$ and using $\sin(2\kappa_1 x)\cos(\kappa_1 x) = \frac{1}{2}[\sin(\kappa_1 x) + \sin(3\kappa_1 x)]$ and $\cos(2\kappa_2 y)\cos(\kappa_2 y) = \frac{1}{2}[\cos(\kappa_2 y) + \cos(3\kappa_2 y)]$:

$$\begin{aligned}\dot\eta\, G_0(\dot\eta\,(-c^*\partial_x\dot\eta)) = {} & \frac{c^*\kappa_1^2}{2}\tanh(2\kappa_1 d)\sin(2\kappa_1 x)\cos(\kappa_1 x)\cos(\kappa_2 y)\\ & + \frac{c^*\kappa_1|\mathbf{k}|}{2}\tanh(2|\mathbf{k}|d)\sin(2\kappa_1 x)\cos(\kappa_1 x)\cos(2\kappa_2 y)\cos(\kappa_2 y).\end{aligned}$$

The first term contributes

$$\frac{c^*\kappa_1^2}{2}\tanh(2\kappa_1 d)\cdot\frac{1}{2} = \frac{c^*\kappa_1^2}{4}\tanh(2\kappa_1 d)$$

to the coefficient of $\sin(\kappa_1 x)\cos(\kappa_2 y)$, while the second term contributes

$$\frac{c^*\kappa_1|\mathbf{k}|}{2}\tanh(2|\mathbf{k}|d)\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{c^*\kappa_1|\mathbf{k}|}{8}\tanh(2|\mathbf{k}|d).$$

Hence the critical-mode part is

$$\left[\frac{c^*\kappa_1^2}{4}\tanh(2\kappa_1 d) + \frac{c^*\kappa_1|\mathbf{k}|}{8}\tanh(2|\mathbf{k}|d)\right]\sin(\kappa_1 x)\cos(\kappa_2 y).$$

Applying the outer G_0 multiplies by $|\mathbf{k}|\tanh(|\mathbf{k}|d)$, and the overall minus sign in T_2 gives

$$T_2 = -\left[\frac{c^*\kappa_1^2}{4}\tanh(2\kappa_1 d) + \frac{c^*\kappa_1|\mathbf{k}|}{8}\tanh(2|\mathbf{k}|d)\right]|\mathbf{k}|\tanh(|\mathbf{k}|d)\,\sin(\kappa_1 x)\cos(\kappa_2 y).$$

All remaining terms lie in noncritical modes and vanish after projection. Therefore,

$$\begin{aligned}\langle T_2, \mathbf{w}_1^*\rangle_{L^2} &= (g+\sigma|\mathbf{k}|^2)\left[-\frac{c^*\kappa_1^2}{4}\tanh(2\kappa_1 d) - \frac{c^*\kappa_1|\mathbf{k}|}{8}\tanh(2|\mathbf{k}|d)\right]|\mathbf{k}|\tanh(|\mathbf{k}|d)\\ &\quad\times\left(\int_0^{2\pi/\kappa_1}\sin^2(\kappa_1 x)\,dx\right)\left(\int_0^{2\pi/\kappa_2}\cos^2(\kappa_2 y)\,dy\right)\\ &= (g+\sigma|\mathbf{k}|^2)\left[-\frac{c^*\kappa_1^2}{4}\tanh(2\kappa_1 d) - \frac{c^*\kappa_1|\mathbf{k}|}{8}\tanh(2|\mathbf{k}|d)\right]|\mathbf{k}|\tanh(|\mathbf{k}|d)\cdot\frac{\pi^2}{\kappa_1\kappa_2}.\end{aligned}$$

Using the dispersion relation $(g + \sigma|\mathbf{k}|^2)|\mathbf{k}|\tanh(|\mathbf{k}|d) = (c^*)^2\kappa_1^2$:

$$\langle T_2, \mathbf{w}_1^* \rangle_{L^2} = -\frac{\pi^2 (c^*)^3 \kappa_1^2}{8\kappa_2}\Big(2\kappa_1 \tanh(2\kappa_1 d) + |\mathbf{k}|\tanh(2|\mathbf{k}|d)\Big). \tag{5.7}$$

Term 3: $T_3 := G_0\Big(\dot\eta\,\nabla\cdot\big(\dot\eta\,\nabla(-G_0^{-1}(c^*\partial_x\dot\eta))\big)\Big)$**.** We compute

$$-G_0^{-1}(c^*\partial_x\dot\eta) = \frac{c^*\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos(\kappa_2 y).$$

Its gradient is

$$\nabla\big(-G_0^{-1}(c^*\partial_x\dot\eta)\big) = \frac{c^*\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\begin{pmatrix}\kappa_1\cos(\kappa_1 x)\cos(\kappa_2 y)\\ -\kappa_2\sin(\kappa_1 x)\sin(\kappa_2 y)\end{pmatrix}.$$

Multiplying by $\dot\eta = \cos(\kappa_1 x)\cos(\kappa_2 y)$:

$$\dot\eta\,\nabla\big(-G_0^{-1}(c^*\partial_x\dot\eta)\big) = \frac{c^*\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\begin{pmatrix}\kappa_1\cos^2(\kappa_1 x)\cos^2(\kappa_2 y)\\ -\kappa_2\sin(\kappa_1 x)\cos(\kappa_1 x)\sin(\kappa_2 y)\cos(\kappa_2 y)\end{pmatrix}.$$

Taking the divergence, the x-component gives

$$\partial_x\left(\frac{c^*\kappa_1^2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\cos^2(\kappa_1 x)\cos^2(\kappa_2 y)\right) = -\frac{c^*\kappa_1^3}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(2\kappa_1 x)\cos^2(\kappa_2 y),$$

and the y-component, using $\partial_y(\sin(\kappa_2 y)\cos(\kappa_2 y)) = \kappa_2\cos(2\kappa_2 y)$, gives

$$\partial_y\left(-\frac{c^*\kappa_1\kappa_2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos(\kappa_1 x)\sin(\kappa_2 y)\cos(\kappa_2 y)\right) = -\frac{c^*\kappa_1\kappa_2^2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos(\kappa_1 x)\cos(2\kappa_2 y).$$

Therefore,

$$\begin{aligned}\nabla\cdot\big(\dot\eta\,\nabla(-G_0^{-1}(c^*\partial_x\dot\eta))\big) = &-\frac{c^*\kappa_1^3}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(2\kappa_1 x)\cos^2(\kappa_2 y)\\ &-\frac{c^*\kappa_1\kappa_2^2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos(\kappa_1 x)\cos(2\kappa_2 y).\end{aligned}$$

Multiplying once more by $\dot\eta = \cos(\kappa_1 x)\cos(\kappa_2 y)$:

$$\begin{aligned}\dot\eta\,\nabla\cdot\big(\dot\eta\,\nabla(-G_0^{-1}(c^*\partial_x\dot\eta))\big) = &-\frac{c^*\kappa_1^3}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(2\kappa_1 x)\cos(\kappa_1 x)\cos^3(\kappa_2 y)\\ &-\frac{c^*\kappa_1\kappa_2^2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos^2(\kappa_1 x)\cos(2\kappa_2 y)\cos(\kappa_2 y).\end{aligned}$$

Using $\sin(2\kappa_1 x)\cos(\kappa_1 x) = \frac{1}{2}[\sin(\kappa_1 x) + \sin(3\kappa_1 x)]$ and $\cos^3(\kappa_2 y) = \frac{1}{4}[3\cos(\kappa_2 y) + \cos(3\kappa_2 y)]$, the first term contributes

$$-\frac{c^*\kappa_1^3}{|\mathbf{k}|\tanh(|\mathbf{k}|d)} \cdot \frac{1}{2} \cdot \frac{3}{4} = -\frac{3c^*\kappa_1^3}{8|\mathbf{k}|\tanh(|\mathbf{k}|d)}$$

to the coefficient of $\sin(\kappa_1 x)\cos(\kappa_2 y)$. Using $\cos^2(\kappa_1 x) = \frac{1}{2}[1 + \cos(2\kappa_1 x)]$ and $\cos(2\kappa_2 y)\cos(\kappa_2 y) = \frac{1}{2}[\cos(\kappa_2 y) + \cos(3\kappa_2 y)]$, only the constant part of $\cos^2(\kappa_1 x)$ contributes to the critical mode, so the second term contributes

$$-\frac{c^*\kappa_1\kappa_2^2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)} \cdot \frac{1}{2} \cdot \frac{1}{2} = -\frac{c^*\kappa_1\kappa_2^2}{4|\mathbf{k}|\tanh(|\mathbf{k}|d)}.$$

Hence the critical-mode part of the inner expression is

$$-\left[\frac{3c^*\kappa_1^3}{8|\mathbf{k}|\tanh(|\mathbf{k}|d)} + \frac{c^*\kappa_1\kappa_2^2}{4|\mathbf{k}|\tanh(|\mathbf{k}|d)}\right]\sin(\kappa_1 x)\cos(\kappa_2 y).$$

Applying the outer G_0 multiplies by $|\mathbf{k}|\tanh(|\mathbf{k}|d)$:

$$T_3 = -\frac{c^*\kappa_1}{8}(3\kappa_1^2 + 2\kappa_2^2)\sin(\kappa_1 x)\cos(\kappa_2 y).$$

All remaining terms lie in noncritical modes and vanish after projection. Therefore,

$$\langle T_3, \mathbf{w}_1^*\rangle_{L^2} = -(g + \sigma|\mathbf{k}|^2)\frac{c^*\kappa_1}{8}(3\kappa_1^2 + 2\kappa_2^2)\left(\frac{\pi}{\kappa_1}\right)\left(\frac{\pi}{\kappa_2}\right),$$

which gives

$$\langle T_3, \mathbf{w}_1^*\rangle_{L^2} = -\frac{\pi^2 c^*}{8\kappa_2}(g + \sigma|\mathbf{k}|^2)\big(3\kappa_1^2 + 2\kappa_2^2\big). \tag{5.8}$$

Term 4: $T_4 := -G_0\big(\nabla\dot\eta\cdot\nabla(-G_0^{-1}(c^*\partial_x\dot\eta))\big)$**.** Computing the dot product $\nabla\dot\eta\cdot\nabla(-G_0^{-1}(c^*\partial_x\dot\eta))$ explicitly and expanding via trigonometric identities, one finds that the result is supported only at modes $(2\kappa_1, 0)$ and $(2\kappa_1, 2\kappa_2)$. Since G_0 acts as a Fourier multiplier and therefore preserves Fourier support, T_4 also lives entirely at these noncritical modes, and, using the same argument as in T_1, orthogonality gives $\langle T_4, \mathbf{w}_1^*\rangle_{L^2} = 0$.

Term 5: $T_5 := \nabla\cdot\big((\dot\eta\,(-c^*\partial_x\dot\eta))\nabla\dot\eta\big)$**.** We have

$$\dot\eta\,(-c^*\partial_x\dot\eta) = \frac{c^*\kappa_1}{4}\sin(2\kappa_1 x) + \frac{c^*\kappa_1}{4}\sin(2\kappa_1 x)\cos(2\kappa_2 y).$$

We compute the divergence of $(\dot\eta\,(-c^*\partial_x\dot\eta))\nabla\dot\eta$ by treating the x- and y-components separately:

$$(\dot\eta\,(-c^*\partial_x\dot\eta))\nabla\dot\eta =: \begin{pmatrix} P \\ Q \end{pmatrix}.$$

For the x-component:

$$\begin{aligned} P &= \left(\frac{c^*\kappa_1}{4}\sin(2\kappa_1 x) + \frac{c^*\kappa_1}{4}\sin(2\kappa_1 x)\cos(2\kappa_2 y)\right)(-\kappa_1\sin(\kappa_1 x)\cos(\kappa_2 y)) \\ &= -\frac{c^*\kappa_1^2}{4}\sin(2\kappa_1 x)\sin(\kappa_1 x)\cos(\kappa_2 y) - \frac{c^*\kappa_1^2}{4}\sin(2\kappa_1 x)\sin(\kappa_1 x)\cos(2\kappa_2 y)\cos(\kappa_2 y). \end{aligned}$$

Using $\sin(2\kappa_1 x)\sin(\kappa_1 x) = \frac{1}{2}[\cos(\kappa_1 x) - \cos(3\kappa_1 x)]$:

$$\begin{aligned} P = &-\frac{c^*\kappa_1^2}{8}[\cos(\kappa_1 x) - \cos(3\kappa_1 x)]\cos(\kappa_2 y) \\ &- \frac{c^*\kappa_1^2}{16}[\cos(\kappa_1 x) - \cos(3\kappa_1 x)][\cos(\kappa_2 y) + \cos(3\kappa_2 y)]. \end{aligned}$$

Differentiating with respect to x:

$$\begin{aligned} \partial_x P = &\frac{c^*\kappa_1^3}{8}[\sin(\kappa_1 x) - 3\sin(3\kappa_1 x)]\cos(\kappa_2 y) \\ &+ \frac{c^*\kappa_1^3}{16}[\sin(\kappa_1 x) - 3\sin(3\kappa_1 x)][\cos(\kappa_2 y) + \cos(3\kappa_2 y)]. \end{aligned}$$

The contribution at the critical mode $\sin(\kappa_1 x)\cos(\kappa_2 y)$ is

$$\frac{c^*\kappa_1^3}{8}\sin(\kappa_1 x)\cos(\kappa_2 y) + \frac{c^*\kappa_1^3}{16}\sin(\kappa_1 x)\cos(\kappa_2 y) = \frac{3c^*\kappa_1^3}{16}\sin(\kappa_1 x)\cos(\kappa_2 y).$$

For the y-component:

$$\begin{aligned} Q &= \left(\frac{c^*\kappa_1}{4}\sin(2\kappa_1 x) + \frac{c^*\kappa_1}{4}\sin(2\kappa_1 x)\cos(2\kappa_2 y)\right)(-\kappa_2\cos(\kappa_1 x)\sin(\kappa_2 y)) \\ &= -\frac{c^*\kappa_1\kappa_2}{4}\sin(2\kappa_1 x)\cos(\kappa_1 x)\sin(\kappa_2 y) \\ &\quad - \frac{c^*\kappa_1\kappa_2}{4}\sin(2\kappa_1 x)\cos(\kappa_1 x)\cos(2\kappa_2 y)\sin(\kappa_2 y). \end{aligned}$$

Using $\sin(2\kappa_1 x)\cos(\kappa_1 x) = \frac{1}{2}[\sin(\kappa_1 x) + \sin(3\kappa_1 x)]$ and $\cos(2\kappa_2 y)\sin(\kappa_2 y) = \frac{1}{2}[\sin(3\kappa_2 y) - \sin(\kappa_2 y)]$:

$$\begin{aligned} Q = &-\frac{c^*\kappa_1\kappa_2}{8}[\sin(\kappa_1 x) + \sin(3\kappa_1 x)]\sin(\kappa_2 y) \\ &- \frac{c^*\kappa_1\kappa_2}{16}[\sin(\kappa_1 x) + \sin(3\kappa_1 x)][\sin(3\kappa_2 y) - \sin(\kappa_2 y)]. \end{aligned}$$

Differentiating with respect to y:

$$\begin{aligned} \partial_y Q = &-\frac{c^*\kappa_1\kappa_2^2}{8}[\sin(\kappa_1 x) + \sin(3\kappa_1 x)]\cos(\kappa_2 y) \\ &- \frac{c^*\kappa_1\kappa_2^2}{16}[\sin(\kappa_1 x) + \sin(3\kappa_1 x)][3\cos(3\kappa_2 y) - \cos(\kappa_2 y)]. \end{aligned}$$

The contribution at the critical mode is

$$-\frac{c^*\kappa_1\kappa_2^2}{8}\sin(\kappa_1 x)\cos(\kappa_2 y) + \frac{c^*\kappa_1\kappa_2^2}{16}\sin(\kappa_1 x)\cos(\kappa_2 y) = -\frac{c^*\kappa_1\kappa_2^2}{16}\sin(\kappa_1 x)\cos(\kappa_2 y).$$

Combining the x- and y-contributions:

$$T_5 = \frac{c^*\kappa_1}{16}(3\kappa_1^2 - \kappa_2^2)\sin(\kappa_1 x)\cos(\kappa_2 y).$$

All remaining terms lie in noncritical modes and vanish after projection. Therefore,

$$\langle T_5, \mathbf{w}_1^* \rangle_{L^2} = (g + \sigma|\mathbf{k}|^2)\frac{c^*\kappa_1}{16}(3\kappa_1^2 - \kappa_2^2)\left(\frac{\pi}{\kappa_1}\right)\left(\frac{\pi}{\kappa_2}\right),$$

which gives

$$\langle T_5, \mathbf{w}_1^* \rangle_{L^2} = \frac{\pi^2 c^*}{16\kappa_2}(g + \sigma|\mathbf{k}|^2)(3\kappa_1^2 - \kappa_2^2). \tag{5.9}$$

Summary of the projection of $\mathcal{F}_1$ onto $\mathbf{w}_1^*$. Among the five terms, T_1 and T_4 are supported entirely on noncritical Fourier modes and vanish by orthogonality, as shown above. The remaining terms T_2, T_3, and T_5 each contribute at the critical mode (κ_1, κ_2). Recalling the factor of 3 from (5.5) and combining (5.7), (5.8), and (5.9):

$$\begin{aligned} \langle \mathcal{F}_1, \mathbf{w}_1^* \rangle &= 3\left[\langle T_2, \mathbf{w}_1^* \rangle_{L^2} + \langle T_3, \mathbf{w}_1^* \rangle_{L^2} + \langle T_5, \mathbf{w}_1^* \rangle_{L^2}\right] \\ &= -\frac{3\pi^2 c^*}{16\kappa_2}\left[2(c^*)^2\kappa_1^2\Big(2\kappa_1\tanh(2\kappa_1 d) + |\mathbf{k}|\tanh(2|\mathbf{k}|d)\Big) + (g + \sigma|\mathbf{k}|^2)(3\kappa_1^2 + 5\kappa_2^2)\right] \neq 0. \end{aligned}$$

The expression in brackets is strictly positive since $c^*, \kappa_1, \kappa_2, g, \sigma > 0$ and both hyperbolic tangent terms are positive, so the inner product is strictly negative and in particular nonzero.

Second component computation

Recall from (5.6) that

$$D^3_{uuu}\mathcal{F}_2[\zeta^*, \zeta^*, \zeta^*] = -3\sigma\nabla \cdot (|\nabla\dot{\eta}|^2\nabla\dot{\eta}) + 6G_0\dot{\Phi}\left(\nabla\dot{\eta} \cdot \nabla\dot{\Phi} + D_\eta G(0)[\dot{\eta}]\dot{\Phi}\right).$$

We project this expression onto the second component of the adjoint kernel generator,

$$\mathbf{w}_2^* = -c^*\kappa_1 \cos(\kappa_1 x)\cos(\kappa_2 y).$$

As before, we use the linearized kinematic relation

$$G_0\dot{\Phi} = -c^*\partial_x\dot{\eta} = c^*\kappa_1 \sin(\kappa_1 x)\cos(\kappa_2 y),$$

together with

$$\dot{\Phi} = \frac{c^*\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)} \sin(\kappa_1 x)\cos(\kappa_2 y).$$

We now treat the cubic terms one at a time.

Curvature term: $C := -3\sigma\nabla \cdot (|\nabla\dot{\eta}|^2\nabla\dot{\eta})$**.** First compute

$$\nabla\dot{\eta} = \begin{pmatrix} -\kappa_1 \sin(\kappa_1 x)\cos(\kappa_2 y) \\ \\ -\kappa_2 \cos(\kappa_1 x)\sin(\kappa_2 y) \end{pmatrix},$$

so that

$$|\nabla\dot{\eta}|^2 = \kappa_1^2 \sin^2(\kappa_1 x)\cos^2(\kappa_2 y) + \kappa_2^2 \cos^2(\kappa_1 x)\sin^2(\kappa_2 y).$$

A direct computation gives

$$\begin{aligned}\nabla \cdot (|\nabla\dot{\eta}|^2\nabla\dot{\eta}) = \Big(&-3\kappa_1^4 \sin^2(\kappa_1 x)\cos^2(\kappa_2 y) + 4\kappa_1^2\kappa_2^2 \sin^2(\kappa_1 x)\sin^2(\kappa_2 y) \\ &- \kappa_1^2\kappa_2^2 \sin^2(\kappa_1 x)\cos^2(\kappa_2 y) - \kappa_1^2\kappa_2^2 \sin^2(\kappa_2 y)\cos^2(\kappa_1 x) \\ &- 3\kappa_2^4 \sin^2(\kappa_2 y)\cos^2(\kappa_1 x)\Big)\cos(\kappa_1 x)\cos(\kappa_2 y).\end{aligned}$$

Using the identities

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta), \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta),$$

and keeping only the contribution at the critical mode $\cos(\kappa_1 x)\cos(\kappa_2 y)$, we obtain

$$\nabla\cdot(|\nabla\dot{\eta}|^2\nabla\dot{\eta}) = -\frac{1}{16}\Big(9\kappa_1^4 + 2\kappa_1^2\kappa_2^2 + 9\kappa_2^4\Big)\cos(\kappa_1 x)\cos(\kappa_2 y) + (\text{noncritical modes}).$$

Therefore

$$C = \frac{3\sigma}{16}\Big(9\kappa_1^4 + 2\kappa_1^2\kappa_2^2 + 9\kappa_2^4\Big)\cos(\kappa_1 x)\cos(\kappa_2 y) + (\text{noncritical modes}).$$

Projecting onto $\mathbf{w}_2^*$ yields

$$\langle C, \mathbf{w}_2^*\rangle_{L^2} = -\frac{3\pi^2 c^*\sigma}{16\kappa_2}\Big(9\kappa_1^4 + 2\kappa_1^2\kappa_2^2 + 9\kappa_2^4\Big). \tag{5.10}$$

Nonlinear fraction, part I: $S_1 := 6G_0\dot{\Phi}\,(\nabla\dot{\eta}\cdot\nabla\dot{\Phi})$**.** Since

$$\dot{\Phi} = \frac{c^*\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos(\kappa_2 y),$$

we have

$$\nabla\dot{\Phi} = \frac{c^*\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\begin{pmatrix}\kappa_1\cos(\kappa_1 x)\cos(\kappa_2 y)\\ \\ -\kappa_2\sin(\kappa_1 x)\sin(\kappa_2 y)\end{pmatrix}.$$

Taking the dot product with $\nabla\dot{\eta}$, we obtain

$$\nabla\dot{\eta}\cdot\nabla\dot{\Phi} = -\frac{c^*\kappa_1^3}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos(\kappa_1 x)\cos^2(\kappa_2 y) + \frac{c^*\kappa_1\kappa_2^2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos(\kappa_1 x)\sin^2(\kappa_2 y).$$

Multiplying by

$$G_0\dot{\Phi} = c^*\kappa_1\sin(\kappa_1 x)\cos(\kappa_2 y),$$

gives

$$\begin{aligned} S_1 &= 6G_0\dot{\Phi}\,(\nabla\dot{\eta}\cdot\nabla\dot{\Phi}) \\ &= -\frac{6(c^*)^2\kappa_1^4}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin^2(\kappa_1 x)\cos(\kappa_1 x)\cos^3(\kappa_2 y) \\ &\quad + \frac{6(c^*)^2\kappa_1^2\kappa_2^2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin^2(\kappa_1 x)\cos(\kappa_1 x)\sin^2(\kappa_2 y)\cos(\kappa_2 y). \end{aligned}$$

After simplifying using trigonometric identities, we see that the contribution at the critical mode is

$$S_1 = -\frac{3(c^*)^2\kappa_1^2}{8|\mathbf{k}|\tanh(|\mathbf{k}|d)}\big(3\kappa_1^2 - \kappa_2^2\big)\cos(\kappa_1 x)\cos(\kappa_2 y).$$

Projecting onto $\mathbf{w}_2^*$ therefore gives

$$\langle S_1, \mathbf{w}_2^*\rangle_{L^2} = \frac{3\pi^2(c^*)^3\kappa_1^2}{8\kappa_2\,|\mathbf{k}|\tanh(|\mathbf{k}|d)}\big(3\kappa_1^2 - \kappa_2^2\big). \tag{5.11}$$

Nonlinear fraction, part II: $S_2 := 6G_0\dot\Phi\, D_\eta G(0)[\dot\eta]\dot\Phi$**.** Using the linearization formula

$$D_\eta G(0)[\dot\eta]\dot\Phi = -G_0(\dot\eta\, G_0\dot\Phi) - \nabla\cdot(\dot\eta\,\nabla\dot\Phi),$$

we compute the two pieces separately. First,

$$\dot\eta\, G_0\dot\Phi = c^*\kappa_1\cos(\kappa_1 x)\sin(\kappa_1 x)\cos^2(\kappa_2 y) = \frac{c^*\kappa_1}{4}\sin(2\kappa_1 x) + \frac{c^*\kappa_1}{4}\sin(2\kappa_1 x)\cos(2\kappa_2 y).$$

Applying G_0, we obtain

$$G_0(\dot\eta\, G_0\dot\Phi) = \frac{c^*\kappa_1\kappa_1}{2}\tanh(2\kappa_1 d)\sin(2\kappa_1 x) + \frac{c^*\kappa_1|\mathbf{k}|}{2}\tanh(2|\mathbf{k}|d)\sin(2\kappa_1 x)\cos(2\kappa_2 y).$$

Next,

$$\nabla\dot\Phi = \frac{c^*\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\begin{pmatrix}\kappa_1\cos(\kappa_1 x)\cos(\kappa_2 y)\\ \\ -\kappa_2\sin(\kappa_1 x)\sin(\kappa_2 y)\end{pmatrix},$$

so

$$\nabla\cdot(\dot\eta\,\nabla\dot\Phi) = -\frac{c^*\kappa_1^3}{2|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(2\kappa_1 x) - \frac{c^*\kappa_1|\mathbf{k}|^2}{2|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(2\kappa_1 x)\cos(2\kappa_2 y).$$

Therefore

$$\begin{aligned} D_\eta G(0)[\dot\eta]\dot\Phi = &\left[\frac{c^*\kappa_1\kappa_1}{2}\tanh(2\kappa_1 d) + \frac{c^*\kappa_1^3}{2|\mathbf{k}|\tanh(|\mathbf{k}|d)}\right]\sin(2\kappa_1 x)\\ &+\left[\frac{c^*\kappa_1|\mathbf{k}|}{2}\tanh(2|\mathbf{k}|d) + \frac{c^*\kappa_1|\mathbf{k}|^2}{2|\mathbf{k}|\tanh(|\mathbf{k}|d)}\right]\sin(2\kappa_1 x)\cos(2\kappa_2 y). \end{aligned}$$

Multiplying by $6G_0\dot{\Phi} = 6c^*\kappa_1 \sin(\kappa_1 x)\cos(\kappa_2 y)$ and keeping only the critical mode $\cos(\kappa_1 x)\cos(\kappa_2 y)$, we obtain

$$S_2 = \frac{3(c^*)^2\kappa_1^2}{4|\mathbf{k}|\tanh(|\mathbf{k}|d)}\Big(3\kappa_1^2+\kappa_2^2+|\mathbf{k}|\tanh(|\mathbf{k}|d)\big(2\kappa_1\tanh(2\kappa_1 d)+|\mathbf{k}|\tanh(2|\mathbf{k}|d)\big)\Big)\cos(\kappa_1 x)\cos(\kappa_2 y).$$

Hence

$$\langle S_2, \mathbf{w}_2^*\rangle_{L^2} = \frac{3\pi^2(c^*)^3\kappa_1^2}{4\kappa_2\,|\mathbf{k}|\tanh(|\mathbf{k}|d)}\Big(3\kappa_1^2+\kappa_2^2+|\mathbf{k}|\tanh(|\mathbf{k}|d)\big(2\kappa_1\tanh(2\kappa_1 d)+|\mathbf{k}|\tanh(2|\mathbf{k}|d)\big)\Big). \tag{5.12}$$

Summary of the projection of $\mathcal{F}_2$ onto $\mathbf{w}_2^*$. Combining (5.10), (5.11), and (5.12), we find

$$\langle \mathcal{F}_2, \mathbf{w}_2^*\rangle_{L^2} = \langle C, \mathbf{w}_2^*\rangle_{L^2} + \langle S_1, \mathbf{w}_2^*\rangle_{L^2} + \langle S_2, \mathbf{w}_2^*\rangle_{L^2}.$$

Substituting (5.10), (5.11), and (5.12):

$$\begin{aligned}\langle \mathcal{F}_2, \mathbf{w}_2^*\rangle_{L^2} = &-\frac{3\pi^2 c^*\sigma}{16\kappa_2}\Big(9\kappa_1^4+2\kappa_1^2\kappa_2^2+9\kappa_2^4\Big)\\ &+\frac{3\pi^2(c^*)^3\kappa_1^2}{8\kappa_2\,|\mathbf{k}|\tanh(|\mathbf{k}|d)}(3\kappa_1^2-\kappa_2^2)\\ &-\frac{3\pi^2(c^*)^3\kappa_1^2}{4\kappa_2\,|\mathbf{k}|\tanh(|\mathbf{k}|d)}\Big[3\kappa_1^2+\kappa_2^2+|\mathbf{k}|\tanh(|\mathbf{k}|d)\big(2\kappa_1\tanh(2\kappa_1 d)+|\mathbf{k}|\tanh(2|\mathbf{k}|d)\big)\Big],\end{aligned}$$

which simplifies to,

$$\begin{aligned}\langle \mathcal{F}_2, \mathbf{w}_2^*\rangle_{L^2} = &-\frac{3\pi^2 c^*\sigma}{16\kappa_2}\Big(9\kappa_1^4+2\kappa_1^2\kappa_2^2+9\kappa_2^4\Big)\\ &-\frac{3\pi^2(c^*)^3\kappa_1^2}{8\kappa_2\,|\mathbf{k}|\tanh(|\mathbf{k}|d)}\Big[3|\mathbf{k}|^2+2|\mathbf{k}|\tanh(|\mathbf{k}|d)\big(2\kappa_1\tanh(2\kappa_1 d)+|\mathbf{k}|\tanh(2|\mathbf{k}|d)\big)\Big].\end{aligned} \tag{5.13}$$

Nonvanishing and sign of (0)

We now combine the computations done thus far with Kielhöfer's formula

$$\ddot{c}(0) = -\frac{1}{3}\frac{\langle D^3_{uuu}\mathcal{F}^*[\zeta^*,\zeta^*,\zeta^*],\mathbf{w}^*\rangle}{\langle D^2_{cu}\mathcal{F}^*[\zeta^*],\mathbf{w}^*\rangle}. \tag{5.14}$$

Recall from the transversality computation (3.9) that

$$\langle D^2_{cu}\mathcal{F}^*[\zeta^*], \mathbf{w}^*\rangle = \langle D_c L_{c^*}[\zeta^*], \mathbf{w}^*\rangle = 2N\kappa_1(g+\sigma|\mathbf{k}|^2) > 0.$$

We decompose the numerator according to the two components of $\mathcal{F}$:

$$\mathcal{N} := \langle D^3_{uuu}\mathcal{F}^*[\zeta^*, \zeta^*, \zeta^*], \mathbf{w}^*\rangle = \langle \mathcal{F}_1, \mathbf{w}_1^*\rangle + \langle \mathcal{F}_2, \mathbf{w}_2^*\rangle. \tag{5.15}$$

Substituting the results of (5.5) and (5.13):

$$\begin{aligned}\mathcal{N} = &-\frac{3\pi^2 c^*}{16\kappa_2}\left[2(c^*)^2\kappa_1^2\Big(2\kappa_1\tanh(2\kappa_1 d) + |\mathbf{k}|\tanh(2|\mathbf{k}|d)\Big) + (g+\sigma|\mathbf{k}|^2)(3\kappa_1^2+5\kappa_2^2)\right]\\ &-\frac{3\pi^2 c^*\sigma}{16\kappa_2}\Big(9\kappa_1^4 + 2\kappa_1^2\kappa_2^2 + 9\kappa_2^4\Big)\\ &-\frac{3\pi^2 (c^*)^3\kappa_1^2}{8\kappa_2\,|\mathbf{k}|\tanh(|\mathbf{k}|d)}\Big[3|\mathbf{k}|^2 + 2|\mathbf{k}|\tanh(|\mathbf{k}|d)\big(2\kappa_1\tanh(2\kappa_1 d) + |\mathbf{k}|\tanh(2|\mathbf{k}|d)\big)\Big].\end{aligned}$$

We simplify by factoring out $-\frac{3\pi^2 c^*}{16\kappa_2}$ from all three terms. For the third term, we use the dispersion relation $\frac{(c^*)^2\kappa_1^2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)} = g+\sigma|\mathbf{k}|^2$ to write

$$\frac{(c^*)^3\kappa_1^2}{|\mathbf{k}|\tanh(|\mathbf{k}|d)} = c^*(g+\sigma|\mathbf{k}|^2),$$

so the third term contributes inside the bracket:

$$2(g+\sigma|\mathbf{k}|^2)\Big[3|\mathbf{k}|^2 + 2|\mathbf{k}|\tanh(|\mathbf{k}|d)\big(2\kappa_1\tanh(2\kappa_1 d) + |\mathbf{k}|\tanh(2|\mathbf{k}|d)\big)\Big].$$

Applying the dispersion relation once more to the hyperbolic part, $2(g+\sigma|\mathbf{k}|^2)|\mathbf{k}|\tanh(|\mathbf{k}|d) = 2(c^*)^2\kappa_1^2$, the hyperbolic contributions from the first and third terms combine to give $6(c^*)^2\kappa_1^2\big(2\kappa_1\tanh(2\kappa_1 d) + |\mathbf{k}|\tanh(2|\mathbf{k}|d)\big)$.

For the polynomial parts, the first and third terms contribute

$$(g+\sigma|\mathbf{k}|^2)(3\kappa_1^2+5\kappa_2^2) + 6(g+\sigma|\mathbf{k}|^2)|\mathbf{k}|^2 = (g+\sigma|\mathbf{k}|^2)(9\kappa_1^2+11\kappa_2^2).$$

Incorporating the surface tension term from the second component, we expand and subtract:

$$
\begin{aligned}
&\big[g+\sigma(\kappa_1^2+\kappa_2^2)\big](9\kappa_1^2+11\kappa_2^2)+\sigma(9\kappa_1^4+2\kappa_1^2\kappa_2^2+9\kappa_2^4)\\
&\quad = g(9\kappa_1^2+11\kappa_2^2)+\sigma(9\kappa_1^4+20\kappa_1^2\kappa_2^2+11\kappa_2^4)+\sigma(9\kappa_1^4+2\kappa_1^2\kappa_2^2+9\kappa_2^4)\\
&\quad = g(9\kappa_1^2+11\kappa_2^2)+\sigma(18\kappa_1^4+22\kappa_1^2\kappa_2^2+20\kappa_2^4).
\end{aligned}
$$

Reassembling, the numerator takes the simplified form

$$
\begin{aligned}
\mathcal{N} = -\frac{3\pi^2 c^*}{16\kappa_2}\Bigg[&6(c^*)^2\kappa_1^2\Big(2\kappa_1\tanh(2\kappa_1 d)+|\mathbf{k}|\tanh(2|\mathbf{k}|d)\Big)\\
&+g(9\kappa_1^2+11\kappa_2^2)+\sigma(18\kappa_1^4+22\kappa_1^2\kappa_2^2+20\kappa_2^4)\Bigg].
\end{aligned}
\tag{5.16}
$$

Since all wave numbers, physical constants $g, \sigma > 0$, and hyperbolic tangent values are strictly positive, every term inside the brackets is strictly positive. Therefore $\mathcal{N} < 0$.

Sign of $\ddot{c}(0)$. Since the denominator in (5.14) is strictly positive and $\mathcal{N} < 0$, we have

$$
\ddot{c}(0) = -\frac{1}{3}\frac{\mathcal{N}}{\langle D_{cu}^2\mathcal{F}^*[\zeta^*], \mathbf{w}^*\rangle} > 0, \tag{5.17}
$$

so the bifurcation is supercritical. In particular, $\ddot{c}(0) \neq 0$, which confirms that the bifurcation is nondegenerate at second order and that the pitchfork condition is satisfied.

BIBLIOGRAPHY

[1] J. Reeder and M. Shinbrot, "Three-dimensional, nonlinear wave interaction in water of constant depth," *Nonlinear Analysis: Theory, Methods Applications*, vol. 5, no. 3, pp. 303–323, 1981. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0362546X81900353

[2] V. E. Zakharov, "Stability of periodic waves of finite amplitude on the surface of a deep fluid," *Journal of Applied Mechanics and Technical Physics*, vol. 9, no. 2, pp. 190–194, 1968.

[3] W. Craig and C. Sulem, "Numerical simulation of gravity waves," *Journal of Computational Physics*, vol. 108, no. 1, pp. 73–83, 1993.

[4] M. G. Crandall and P. H. Rabinowitz, "Bifurcation from simple eigenvalues," *Journal of Functional Analysis*, vol. 8, no. 2, pp. 321–340, 1971.

[5] A. I. Nekrasov, "On steady waves," *Izvestiya Ivanovo-Voznesensk Polytechnic Institute*, 1921, in Russian.

[6] T. Levi-Civita, "Détermination rigoureuse des ondes permanentes d'ampleur finie," *Mathematische Annalen*, vol. 93, no. 1, pp. 264–314, 1925.

[7] D. J. Struik, "Détermination rigoureuse des ondes irrotationelles périodiques dans un canal à profondeur finie," *Mathematische Annalen*, vol. 95, no. 1, pp. 595–634, 1926.

[8] G. Iooss, P. I. Plotnikov, and J. F. Toland, "Standing waves on an infinitely deep perfect fluid under gravity," *Archive for Rational Mechanics and Analysis*, vol. 177, no. 3, pp. 367–478, 2005.

[9] R. S. Hamilton, *The Inverse Function Theorem of Nash and Moser*, ser. Bulletin (New Series). American Mathematical Society, 1982, vol. 7, no. 1.

[10] S. Haziot, V. Hur, W. Strauss, J. Toland, E. Wahlén, S. Walsh, and M. Wheeler, "Traveling water waves—the ebb and flow of two centuries," *Quarterly of applied mathematics*, vol. 80, no. 2, pp. 317–401, 2022.

[11] D. Lannes, *The water waves problem: mathematical analysis and asymptotics.* American Mathematical Soc., 2013, vol. 188.

[12] F. Rousset and N. Tzvetkov, "Transverse instability of the line solitary water-waves," *arXiv preprint arXiv:0906.2487*, 2009.

[13] T. Kato, *Perturbation theory for linear operators*, ser. Classics in Mathematics. Springer-Verlag, Berlin, 1995, reprint of the 1980 edition.

[14] H. Kielhöfer, *Bifurcation theory.* Springer, 2012.