Symmetric Diamond Waves Revisited: Existence via the Crandall–Rabinowitz Theorem

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Outline

Water waves

Diamond waves

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Diamond waves

- Water waves model a free surface of fluid acted on by gravity and surface tension.
- **Diamond waves** arise from two symmetric wave trains of equal speed, producing a doubly periodic surface pattern.

Brief history

2D — planar Stokes waves:

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This thesis

Re-derive existence via the **Crandall–Rabinowitz theorem** and identify the type of bifurcation (new).

Governing equations

The fluid is **inviscid, incompressible, and irrotational**, so $u = \nabla\phi$ for a scalar potential ϕ.

Interior:

$$\Delta\phi = 0 \quad \text{in } \Omega(t).$$

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Free surface $S(t) = \{z = \eta(\mathbf{x}, t)\}$ — two conditions:

- Kinematic: $\partial_t\eta - \partial_z\phi + \nabla'\eta \cdot \nabla'\phi = 0.$
- Dynamic: $\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g\eta - \sigma H(\eta) = 0$, where $H(\eta)$ is twice the mean curvature and $\sigma > 0$.

Travelling waves

Seek solutions steady in a frame moving at speed c in the x-direction:

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Setting $\partial_t \mapsto -c\,\partial_x$ reduces the system to a **steady free-boundary problem**. We have

$$\begin{aligned} \Delta\phi &= 0 \quad \text{in } \Omega, \\ \partial_z \phi &= 0 \quad \text{on } z = -d, \\ -c\partial_x \eta - \partial_z \phi + \nabla' \eta \cdot \nabla' \phi &= 0 \quad \text{on } S, \\ -c\partial_x \phi + \frac{1}{2}|\nabla\phi|^2 + g\eta - \sigma H(\eta) &= 0 \quad \text{on } S, \end{aligned}$$

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The **Dirichlet–Neumann operator** maps Φ to the scaled outward normal derivative of ϕ at the free surface:

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$G(\eta)$ encodes all interior fluid dynamics in a single surface operator.

Properties of $G(\eta)$

(i) Analyticity. $\eta \mapsto G(\eta)$ is analytic on Sobolev spaces (Craig–Nicholls). Guarantees $\mathcal{F}$ is smooth near the trivial solution.

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$$G_0 \, e^{i\mathbf{k}\cdot\mathbf{x}} = |\mathbf{k}| \tanh(|\mathbf{k}|d) \, e^{i\mathbf{k}\cdot\mathbf{x}}.$$

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(iii) First variation at $\eta = 0$:

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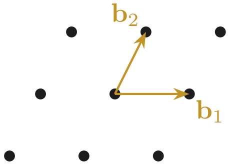
(iv) Linearization at the trivial solution. $D_\eta G(0)[\dot\eta]$ is evaluated at $\Phi = 0$. We

get

$$D_{(\eta,\Phi)}[G(\eta)\Phi]\big|_{(0,0)}(\dot\eta, \dot\Phi) = G_0\dot\Phi.$$

Spatial lattice



Lattice Λ

Solutions periodic with respect to

$$\Lambda := \{m_1\mathbf{b}_1 + m_2\mathbf{b}_2 : m_1, m_2 \in \mathbb{Z}\}.$$

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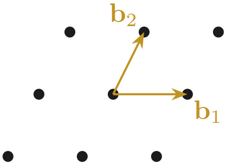

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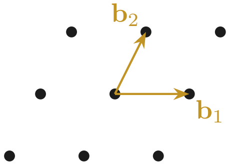

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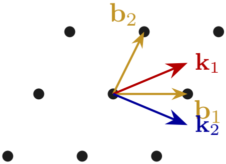
Each $\mathbf{k} \in \Lambda^*$ corresponds to the mode $e^{i\mathbf{k}\cdot\mathbf{x}}$. Differential operators act *diagonally* on individual modes.

Diamond pattern



Wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$ in Λ^*

The diamond pattern arises when there exist

$$\mathbf{k}_1 = (\kappa_1, \kappa_2), \quad \mathbf{k}_2 = (\kappa_1, -\kappa_2) \in \Lambda^*$$

with $|\mathbf{k}_1| = |\mathbf{k}_2|$ and angle $\theta \in (0, \pi/2)$.

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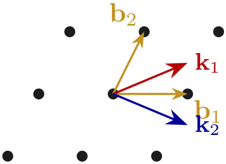

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The pattern is the superposition of two wave trains at angles $\pm\theta$ relative to the x-axis, each with the same wave speed.

Symmetric subspace

To isolate the diamond pattern, impose parity conditions:

- η even in both x and y,
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For $s > 2$, define the solution and target spaces

$$X := H^s_{\mathsf{ee}}(\Lambda) \times H^s_{\mathsf{oe}}(\Lambda), \qquad Y := H^{s-1}_{\mathsf{oe}}(\Lambda) \times H^{s-2}_{\mathsf{ee}}(\Lambda).$$

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These conditions are compatible with the equations: the kinematic equation maps $H^s_{\mathsf{oe}} \to H^{s-1}_{\mathsf{oe}}$ and the dynamic equation maps $H^s_{\mathsf{ee}} \to H^{s-2}_{\mathsf{ee}}$.

Operator equation

In the travelling-wave frame, define

$$\mathcal{F}(\eta, \Phi\,, c) := \begin{pmatrix} -c\,\partial_x \eta + G(\eta)\Phi \\ -c\,\partial_x \Phi + \frac{1}{2}|\nabla' \Phi|^2 - \dfrac{(\nabla' \eta \cdot \nabla' \Phi + G(\eta)\Phi)^2}{2(1 + |\nabla' \eta|^2)} + g\eta - \sigma H(\eta) \end{pmatrix}.$$

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$$\mathcal{F}(\eta, \Phi\,, c) = 0, \qquad \mathcal{F} : X \times \mathbb{R} \to Y.$$

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The trivial solution $(\eta,\Phi) = (0,0)$ satisfies $\mathcal{F} = 0$ for every c. We look for solutions **bifurcating from this trivial branch**.

Main theorem

Theorem (Existence of Diamond Waves)

Let $\lambda = \frac{1}{|\mathbf{k}_1|}\sqrt{g/\sigma}$ *and* $\mathbf{k}_1 = |\mathbf{k}_1|(\cos\theta, \sin\theta)$*. Assume* $(\lambda, \theta) \notin \mathscr{M}$*, where* $\mathscr{M}$ *is a nowhere-dense set. Then there exists a family of* Λ*-periodic smooth solutions* $(\eta, \Phi, c)(\varepsilon)$ *to* $\mathcal{F} = 0$ *bifurcating at* $(0, 0, c^*)$*, where*

$$c^* = c(\mathbf{k}_1) = c(\mathbf{k}_2), \qquad c(\mathbf{k})^2 = \frac{(g + \sigma|\mathbf{k}|^2)\,|\mathbf{k}|\tanh(|\mathbf{k}|d)}{\kappa_1^2}.$$

The free surface satisfies $\eta(\varepsilon) = \varepsilon\cos(\kappa_1 x)\cos(\kappa_2 y) + O(\varepsilon^2)$*.*

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Proof: verify the hypotheses of the Crandall–Rabinowitz theorem for $\mathcal{F}$ at $c = c^*$.

Fredholm operators and Fréchet derivatives

Fredholm operators. A bounded linear $\mathcal{L} : X \to Y$ is *Fredholm* if $\dim \mathscr{N}(\mathcal{L}) < \infty$, $\operatorname{codim} \mathscr{R}(\mathcal{L}) < \infty$, and $\mathscr{R}(\mathcal{L})$ is closed. Its index is

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Fréchet derivative. F is *Fréchet differentiable* at $x \in X$ if there exists a bounded linear operator $\mathcal{L}^x : X \to Y$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|F(x+h) - F(x) - \mathcal{L}^x h\|_Y}{\|h\|_X} = 0.$$

Crandall–Rabinowitz theorem

Theorem (Crandall–Rabinowitz)

Let $\mathcal{F} : \mathbb{R} \times X \to Y$ *be* C^2*. Suppose at* λ_0*:*

(i) $\mathcal{F}(\lambda, 0) = 0$ *for all* $\lambda \in \mathbb{R}$*;*

(ii) $D_x\mathcal{F}(\lambda_0, 0)$ *is Fredholm of index zero with* $\dim \mathscr{N}(D_x\mathcal{F}(\lambda_0, 0)) = 1$*, say* $\mathscr{N}(D_x\mathcal{F}(\lambda_0, 0)) = \mathrm{span}\{u_0\}$*;*

(iii) $D_\lambda D_x\mathcal{F}(\lambda_0, 0)\, u_0 \notin \mathscr{R}(D_x\mathcal{F}(\lambda_0, 0))$.

Then $(\lambda_0, 0)$ *is a bifurcation point: there exists a* C^1 *curve* $\mathscr{C} = \{(\lambda(s), x(s)) : s \in (-\varepsilon, \varepsilon)\}$ *with* $\mathcal{F}(\lambda(s), x(s)) = 0$*,* $x(s) \neq 0$ *for* $s \neq 0$*, and* $\lambda(s) = \lambda_0 + O(s)$*,* $x(s) = su_0 + O(s^2)$*.*

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$\lambda_0 \longleftrightarrow$ bifurcation point c^*

$u_0 \longleftrightarrow$ kernel generator ζ^*

Proof overview

Step	Content	Verifies
1	$\mathcal{F} : X \times \mathbb{R} \to Y$ well-defined and smooth	(prerequisite)
2	Linearized operator L_c and dispersion relation	(setup)
3	One-dimensional kernel	(i)
4	L_{c^*} is Fredholm of index zero	(ii)
5	Transversality	(iii)

Smoothness and mapping properties

For $s > 2$, $\mathcal{F} : X \times \mathbb{R} \to Y$ is well-defined and smooth.

For $\mathcal{F}_1 = -c\,\partial_x \eta + G(\eta)\Phi \in H^{s-1}_{\mathbf{oe}}$:

- $\eta \in H^s_{\mathrm{ee}} \Rightarrow \partial_x \eta \in H^{s-1}_{\mathrm{oe}}$.
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For $\mathcal{F}_2 \in H^{s-2}_{\textbf{ee}}$ — each term is even-even:

- $c\partial_x\Phi$: $\Phi \in H^s_{\mathsf{oe}}$ so $\partial_x\Phi$ is even-even.
- $|\nabla'\Phi|^2$: $\partial_x\Phi$ is ee, $\partial_y\Phi$ is oo; both squares are ee.
- $\sigma H(\eta)$: $\nabla'\eta$ has components oe and eo; divergence of the normalized gradient is ee.
- Nonlinear fraction: numerator base is oe, squaring gives ee; denominator is ee. ✓

Linearized operator

The linearized operator at $(0, 0\,, c)$ is

$$L_c \begin{pmatrix} \dot\eta \\ \dot\Phi \end{pmatrix} = \begin{pmatrix} -c\,\partial_x \dot\eta + G_0 \dot\Phi \\ (g - \sigma\Delta)\dot\eta - c\,\partial_x \dot\Phi \end{pmatrix}.$$

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L_c acts *diagonally* on Fourier modes — determined by its action at each mode separately.

Dispersion relation

On mode $\mathbf{k} = (\kappa_1, \kappa_2) \in \Lambda^*$, the equation $L_c(\dot{\eta}, \dot{\Phi})^T = 0$ reduces to the 2×2 system

$$M_{\mathbf{k}}(c) \begin{pmatrix} \hat{\eta}_{\mathbf{k}} \\ \hat{\Phi}_{\mathbf{k}} \end{pmatrix} = \mathbf{0}, \qquad M_{\mathbf{k}}(c) = \begin{pmatrix} -c\kappa_1 & |\mathbf{k}| \tanh(|\mathbf{k}|d) \\ -(g + \sigma|\mathbf{k}|^2) & c\kappa_1 \end{pmatrix}.$$

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A nontrivial solution exists if and only if $\det M_{\mathbf{k}}(c) = 0$:

$$\boxed{D_0(\mathbf{c}, \mathbf{k}) := g + \sigma|\mathbf{k}|^2 - \frac{(\mathbf{c} \cdot \mathbf{k})^2}{|\mathbf{k}|} \coth(|\mathbf{k}|d) = 0.}$$

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Solving for c gives the wave speed

$$c(\mathbf{k})^2 = \frac{(g + \sigma|\mathbf{k}|^2)\, |\mathbf{k}| \tanh(|\mathbf{k}|d)}{\kappa_1^2}$$

and $\dim \mathscr{N}(L_c) = \#\{\mathbf{k} \in \Lambda^* : D_0(\mathbf{c}, \mathbf{k}) = 0\}$.

One-dimensional kernel

When $(\lambda, \theta) \notin \mathscr{M}$, only $\{\pm\mathbf{k}_1, \pm\mathbf{k}_2\}$ satisfy $D_0(c^*, \mathbf{k}) = 0$, so $\dim \mathscr{N}(L_{c^*}) = 4$ on the full space.

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Normalize $\hat{\eta}_{\mathbf{k}} = 1$:

$$\zeta^* = \begin{pmatrix} \cos(\kappa_1 x)\cos(\kappa_2 y) \\ \dfrac{c^*\kappa_1}{|\mathbf{k}|\tanh(|\mathbf{k}|d)}\sin(\kappa_1 x)\cos(\kappa_2 y) \end{pmatrix}, \qquad \mathscr{N}(L_{c^*}) = \operatorname{span}\{\zeta^*\}. \quad \checkmark$$

Fredholm property and range characterisation

Claim: $\dim \mathscr{N}(L_{c^*}) = 1$, $\operatorname{codim} \mathscr{R}(L_{c^*}) = 1$, and $\operatorname{ind}(L_{c^*}) = 0$.

Fredholm property and range characterisation

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$$(f_1, f_2)^T \in \mathscr{R}(L_{c^*}) \text{ if and only if } \langle (f_1, f_2)^T, \mathbf{w}^* \rangle_{L^2} = 0,$$

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Closed range. If $f^{(n)} \in \mathscr{R}(L_{c^*})$ and $f^{(n)} \to f$ in Y, the solvability condition passes to the limit by continuity of Fourier coefficients. Hence, $\operatorname{ind}(L_{c^*}) = 0$. ✓

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Taking the L^2 inner product with $\mathbf{w}^*$ and using the dispersion relation:

$$\langle D_c L_{c^*}[\zeta^*], \mathbf{w}^* \rangle = 2N\kappa_1(g + \sigma|\mathbf{k}|^2) > 0, \quad \checkmark$$

where $N = \|\sin(\kappa_1 x)\cos(\kappa_2 y)\|_{L^2}^2 > 0$.

Conclusion

Hypothesis	**Result**	
(i) Fredholm index zero	$\operatorname{ind}(L_{c^*}) = 0$	✓
(ii) One-dimensional kernel	$\mathscr{N}(L_{c^*}) = \operatorname{span}\{\zeta^*\}$	✓
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All hypotheses of the Crandall–Rabinowitz theorem are satisfied.

The theorem delivers a C^1 curve $(\eta(\varepsilon), \Phi(\varepsilon), c(\varepsilon))$ bifurcating from $(0, 0, c^*)$.

Theorem 1 is proved.

What kind of bifurcation?

Define the linearized operator along the branch:

$$L(s) := D_u\mathcal{F}(u(s), c(s)), \qquad u(s) = (\eta(s), \Phi(s)).$$

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By Kato's perturbation theory, $L(s)$ has a simple eigenvalue $\mu_0(s)$ with $\mu_0(0) = 0$. Invertibility is equivalent to $\mu_0(s) \neq 0$ for small $s \neq 0$.

Lyapunov–Schmidt reduction

The equation $\mathcal{F} = 0$ reduces to a scalar bifurcation equation

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$$\dot{c}(0) = -\frac{\langle D^2_{uu}\mathcal{F}(0, c^*)[\zeta^*, \zeta^*], \mathbf{w}^* \rangle}{2\langle D_c L_{c^*}[\zeta^*], \mathbf{w}^* \rangle}.$$

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Numerator $= 0$: quadratic terms in $\mathcal{F}$ are even-even, but $\mathbf{w}^*$ is odd-even — so the inner product vanishes by parity.

Denominator $\neq 0$: transversality condition.

So, $\dot{c}(0) = 0$. Transcritical bifurcation is ruled out. We suspect a **pitchfork**, and go to second order.

Second-order analysis

Since $\dot{c}(0) = 0$, Kielhöfer's formula gives:

$$\ddot{c}(0) = -\frac{1}{3}\frac{\langle D^3_{uuu}\mathcal{F}(0, c^*)[\zeta^*, \zeta^*, \zeta^*], \mathbf{w}^*\rangle}{\langle D_c L_{c^*}[\zeta^*], \mathbf{w}^*\rangle}.$$

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- Numerator: shown in next slide.

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Once $\ddot{c}(0) \neq 0$, Kielhöfer gives $\ddot{\mu}_0(0) \neq 0$, and

$$\mu_0(s) = \tfrac{1}{2}\ddot{\mu}_0(0)\, s^2 + O(s^3) \neq 0 \quad \text{for } 0 < |s| \ll 1.$$

So $L(s)$ is invertible.

Cubic computation

Evaluating $\mathcal{N} := \langle D^3_{uuu}\mathcal{F}(0,c^*)[\zeta^*,\zeta^*,\zeta^*],\mathbf{w}^*\rangle$:

- Expand $G(\eta)$ to second order; use the first-variation formula for $D_\eta G(0)$ to simply computations.
- Use $G_0\dot{\Phi} = -c^*\partial_x\dot{\eta}$.
- Many terms vanish by orthogonality; examine only the critical mode.

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- Many terms vanish by orthogonality; examine only the critical mode.

Result:

$$\mathcal{N} = -\frac{3\pi^2 c^*}{16\kappa_2}\Big[6(c^*)^2\kappa_1^2\big(2\kappa_1\tanh(2\kappa_1 d) + |\mathbf{k}|\tanh(2|\mathbf{k}|d)\big) + g(9\kappa_1^2 + 11\kappa_2^2) \\ + \sigma(18\kappa_1^4 + 22\kappa_1^2\kappa_2^2 + 20\kappa_2^4)\Big].$$

Every term in the bracket is strictly positive, so $\mathcal{N}$ is nonzero.

Sign of $\ddot{c}(0)$

$$\ddot{c}(0) = -\frac{1}{3}\,\frac{\overbrace{\mathcal{N}}^{<\,0}}{\underbrace{\langle D_c L_{c^*}[\zeta^*], \mathbf{w}^*\rangle}_{>\,0}} \;>\; 0.$$

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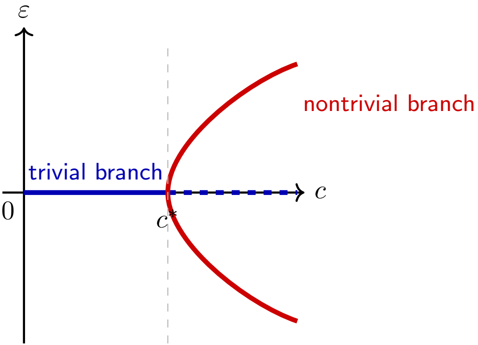
Supercritical pitchfork bifurcation



Solid: stable flat water. Dashed: unstable.
Red: nontrivial branch.

- Flat water is stable for $c < c^*$; loses stability at c^*.
- Diamond waves emerge for $c > c^*$.

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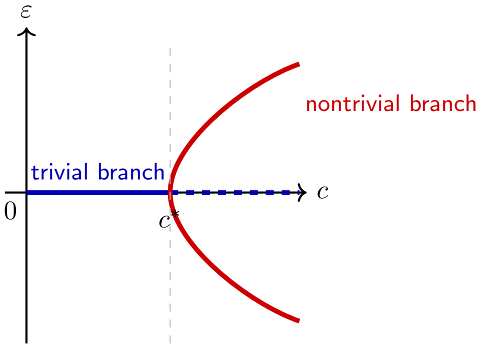

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Contrasts with a **subcritical** bifurcation, where the nontrivial branch bends back to $c < c^*$.

Summary

1. **Reformulation.** 3D capillary-gravity water wave problem as $\mathcal{F}(\eta, \Phi, c) = 0$ via the Zakharov–Craig–Sulem reduction.
2. **Existence.** Diamond waves constructed via Crandall–Rabinowitz: Fredholm index zero, one-dimensional kernel, transversality.
3. **Supercritical pitchfork.** Sign $\ddot{c}(0) > 0$ established via an explicit third-order Fréchet derivative computation.

Thank you!

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